NEW CASES OF p-ADIC UNIFORMIZATION

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To G. Laumon on his 60th birthday

Contents

1. Introduction

The subject matter of p-adic uniformization of Shimura varieties starts with Cherednik's paper [6] in 1976, although a more thorough historical account would certainly involve at least the names of Mumford and Tate. Cherednik's theorem states that the Shimura curve associated to a quaternion algebra B over a totally real field F which is split at precisely one archimedean place v of F (and ramified at all other archimedean places), and is ramified at a non-archimedean place w of residue characteristic p admits p-adic uniformization by the Drinfeld halfplane associated to F_w, provided that the level structure is prime to p. In adelic terms, this theorem may be formulated more precisely as follows.

Let C be an open compact subgroup of $(B \otimes_F \mathbb{A}_F^\infty)^\times$ of the form

$$C = C^w \cdot C_w,$$

where $C_w \subset (B \otimes_F F_w)^\times$ is maximal compact and $C^w \subset (B \otimes_F \mathbb{A}_F^{\infty,w})^\times$. Let $\mathcal{S}_C$ be the associated Shimura curve. It has a canonical model over F and its set of complex points, for the F-algebra structure on $\mathbb{C}$ given by v, has a complex uniformization

$$\mathcal{S}_C(\mathbb{C}) = B^\times \backslash \big[\mathbf{X} \times (B \otimes_F \mathbb{A}_F^\infty)^\times / C\big],$$

where $\mathbf{X} = \mathbb{C} \smallsetminus \mathbb{R}$, which is acted on by $(B \otimes_F F_\mathbb{R})^\times$ via a fixed isomorphism $B_v^\times \simeq \mathrm{GL}_2(\mathbb{R})$.

Cherednik's theorem states that, after extending scalars from F to $\bar{F}_w$, there is an isomorphism of algebraic curves over $\bar{F}_w$,

$$\mathcal{S}_C \otimes_F \bar{F}_w \simeq \big(\bar{B}^\times \backslash \big[\Omega^2_{F_w} \times (B \otimes_F \mathbb{A}_F^\infty)^\times / C\big)\big] \otimes_{F_w} \bar{F}_w, \tag{1.1}$$

where $\bar{B}$ is the quaternion algebra over F, with the same invariants as B, except at v and w, where they are interchanged. Here $\Omega^2_{F_w}$ is the rigid-analytic space $\mathbb{P}^1_{F_w} \smallsetminus \mathbb{P}^1(F_w)$ over F_w (*Drinfeld's halfspace*). This isomorphism is to be interpreted as follows.

The rigid-analytic space $\bar{B}^\times \backslash \big[\Omega^2_{F_w} \times (B \otimes_F \mathbb{A}_F^\infty)^\times / C\big)\big]$ corresponds to a unique projective algebraic curve over Spec F_w under the GAGA functor. In the right hand side of

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(1.1), we implicitly replace the rigid-analytic space by this projective scheme; extending scalars, we obtain a projective algebraic curve over $\bar{F}_w$. The statement of Cherednik's theorem is that there exists an isomorphism between these two algebraic curves over $\bar{F}_w$.

Drinfeld [8] gave a moduli-theoretic proof of Cherednik's theorem in the special case $F = \mathbb{Q}$. Furthermore, he proved an 'integral version' of this theorem which has the original version as a corollary. In his formulation appears the formal scheme $\widehat{\Omega}^2_{F_w}$ over Spec O_{F_w}, with "generic fiber" equal to $\Omega^2_{F_w}$, defined by Mumford, Deligne and Drinfeld. In particular, he interpreted the formal scheme $\widehat{\Omega}^2_{F_w}$, and its higher-dimensional versions $\widehat{\Omega}^n_{F_w}$ as formal moduli spaces of *special formal* O_{B_w}*-modules*, where B_w is the central division algebra over F_w with invariant $1/n$.

This integral uniformization theorem was generalized to higher-dimensional cases in [26]. In these cases, one uniformizes Shimura varieties associated to certain unitary groups over a totally real field F which at the archimedean places have signature $(1, n-1)$ at one place v, and signature $(0, n)$ at all others, and such that the associated CM-field K has two distinct places over the p-adic place w of F. (One has to be much, much more specific to force p-adic uniformization, cf. loc. cit. pp. 298–315). Using these methods, Boutot and Zink [3] have given a conceptual proof of Cherednik's theorem for general totally real fields, and constructed at the same time integral models for the corresponding Shimura varieties. Such integral models were also constructed for general Shimura curves by Carayol [4]. In this context also falls the work of Varshavsky [28], which concerns the p-adic uniformization of Shimura varieties associated to similar unitary groups, again where the p-adic place w splits in K (but not the construction of integral models).

In this paper, we give a new (very restricted) class of Shimura varieties which admit p-adic uniformization. For this class we prove p-adic uniformization for their generic fibers and, for some members in this class, p-adic uniformization even for their integral models. The simplest example is the following. Let K be an imaginary quadratic field, and let V be a hermitian vector space of dimension 2 over K of signature $(1, 1)$. Let $G = \mathrm{GU}(V)$ be the group of unitary similitudes of V. For $C \subset G(\mathbb{A}^\infty)$ an open compact subgroup, there is a Shimura variety Sh_C with canonical model over $\mathbb{Q}$ whose complex points are given by

$$\mathrm{Sh}_C(\mathbb{C}) \simeq G(\mathbb{Q})\backslash[\mathbf{X} \times G(\mathbb{A}^\infty)/C],$$

where again $\mathbf{X} = \mathbb{C} \smallsetminus \mathbb{R}$, which is acted on by $G(\mathbb{R})$ via a fixed isomorphism $G_{\mathrm{ad}}(\mathbb{R}) \simeq \mathrm{PGL}_2(\mathbb{R})$.

Let p be a prime that does *not* split in K. For technical reasons we assume that $p \neq 2$ if p is ramified in K. Suppose that the local hermitian space $V \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is *anisotropic* and that C has the form $C = C^p \cdot C_p$, where C_p is the *unique* maximal compact subgroup of $G(\mathbb{Q}_p)$. Let $\bar{V}$ be the hermitian space over K which is positive definite, split at p, and locally coincides with V at all places $\neq \infty, p$, and let $I = \mathrm{GU}(\bar{V})$ be the corresponding group of unitary similitudes. Then there is an identification of the adjoint group $I_{\mathrm{ad}}(\mathbb{Q}_p)$ with $\mathrm{PGL}_2(\mathbb{Q}_p)$ and an action of $I(\mathbb{Q})$ on $G(\mathbb{A}^\infty)/C$.

Theorem 1.1. *There is an isomorphism of algebraic curves over the completion of the maximal unramified extension* $\breve{\mathbb{Q}}_p$ *of* $\mathbb{Q}_p$*,*

$$\mathrm{Sh}_C \otimes_{\mathbb{Q}} \breve{\mathbb{Q}}_p \simeq \big(I(\mathbb{Q})\backslash[\Omega^2_{\mathbb{Q}_p} \times G(\mathbb{A}^\infty)/C]\big) \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p$$

We also prove a generalization to higher dimensions, cf. Corollary 7.3. The statement in Theorem 1.1 looks formally very similar to Cherednik's. However, the Shimura varieties here are not the same (although the corresponding algebraic groups have isomorphic adjoint groups); in particular, our Shimura varieties represent a moduli problem

of abelian varieties with additional structure, whereas this is not true of Cherednik's Shimura varieties if $F \neq \mathbb{Q}$.

By extending the moduli problem integrally, we obtain integral models of these Shimura varieties. This gives us the possibility of formulating and proving an 'integral' version of this theorem, which is also valid in higher dimensions. Behind this integral version of Theorem 1.1 is our interpretation of the Drinfeld formal halfplane $\widehat{\Omega}^2_F$, for a p-adic local field F and a quadratic extension K of F, as the formal moduli space of polarized two-dimensional O_K-modules of Picard type, established in a previous paper [16]. This interpretation is valid for any p-adic field F (with, as usual, a caveat when $p = 2$). It is remarkable that, for the p-adic uniformization of our class of Shimura varieties, only the case $F = \mathbb{Q}_p$ is relevant. This can be traced to the fact that, on the one hand the formal moduli problem for $\widehat{\Omega}^2_F$ imposes that O_F acts through the structure morphism on the Lie algebra of the formal groups appearing, but that, on the other hand, the *Kottwitz condition* for the action of O_F on the Lie algebra of the relevant abelian varieties requires that the Lie algebra be a free $O_F \otimes \mathcal{O}_S$-algebra, locally on S.

A simplified version of our main theorem about integral uniformization is as follows[1]. Let K be a CM quadratic extension of a totally real field F of degree d over $\mathbb{Q}$. Let p be a prime that decomposes completely in F and such that each prime divisor $\mathbf{p}$ of p in F is inert or ramified in K. Assume that if $p = 2$, then no $\mathbf{p}$ is ramified in K. Let V be a hermitian vector space of dimension 2 over K with signature $(1, 1)$ at every archimedean place of F. We also assume that $\mathrm{inv}_{\mathbf{p}}(V) = -1$ for all $\mathbf{p}|p$. Let G be the group of unitary similitudes of V with multiplier in $\mathbb{Q}^\times$. Let C^p be an open compact subgroup of $G(\mathbb{A}^{\infty,p})$, and let $C = C^p \cdot C_p$, where C_p is the unique maximal compact subgroup of $G(\mathbb{Q}_p)$. Let Sh_C be the canonical model of the corresponding Shimura variety. It is a projective variety of dimension d defined over $\mathbb{Q}$.

Let $\mathcal{M}_{r,h,V}(C^p)$ be the model of Sh_C over $\mathbb{Z}_{(p)}$ which parametrizes *almost principal CM-triples* (A, ι, λ) *of generalized CM-type* (r, h) with level-C^p-structure, where $r_\varphi = 1$, for all complex embeddings φ of K, and where the function h describes the kernel of the polarization λ (which has to satisfy the compatibility condition of Proposition 4.2). In particular, we demand that the localization of the kernel of the polarization λ at any $\mathbf{p}|p$ satisfies

$$\begin{aligned} p \cdot \big(\mathrm{Ker}\ \lambda\big)_{\mathbf{p}} &= (0), \\ |(\mathrm{Ker}\ \lambda)_{\mathbf{p}}| &= \begin{cases} p^2 & \text{when } \mathbf{p}|p \text{ is unramified,} \\ 1 & \text{when } \mathbf{p}|p \text{ is ramified.} \end{cases} \end{aligned} \tag{1.2}$$

In addition, for each place v of F, the invariant $\mathrm{inv}_v(A, \iota, \lambda)$, defined in section 3, is required to coincide with the invariant $\mathrm{inv}_v(V)$ of the hermitian space $V-$ see section 3 for the precise definitions. We denote by $\mathcal{M}_{r,h,V}(C^p)^\wedge$ the completion of this model along its special fiber.

Theorem 1.2. *There is a $G(\mathbb{A}^{\infty,p})$-equivariant isomorphism of p-adic formal schemes*

$$\mathcal{M}_{r,h,V}(C^p)^\wedge \times_{\mathrm{Spf}\,\mathbb{Z}_p} \mathrm{Spf}\,\breve{\mathbb{Z}}_p \simeq I(\mathbb{Q})\backslash\big[\big((\widehat{\Omega}^2_{\mathbb{Q}_p})^d \times_{\mathrm{Spf}\,\mathbb{Z}_p} \mathrm{Spf}\,\breve{\mathbb{Z}}_p\big) \times G(\mathbb{A}^\infty)/C\big]\,.$$

Here $I(\mathbb{Q})$ is the group of $\mathbb{Q}$-rational points of the inner form I of G such that $I_{\mathrm{ad}}(\mathbb{R})$ is compact, $I_{\mathrm{ad}}(\mathbb{Q}_p) \simeq \mathrm{PGL}_2(\mathbb{Q}_p)^d$, and $I(\mathbb{A}^{\infty,p}) \simeq G(\mathbb{A}^{\infty,p})$.

The natural descent datum on the LHS induces on the RHS the natural descent datum on the first factor multiplied with the translation action of $(1, t)$ on $G(\mathbb{A}^\infty)/C = G(\mathbb{A}^{\infty,p})/C^p \times G(\mathbb{Q}_p)/C_p$, where $t \in G(\mathbb{Q}_p)$ is any element with $\mathrm{ord}_p\, c(t) = 1$ *for $c : G(\mathbb{Q}_p) \to \mathbb{Q}_p^\times$ the scale homomorphism.*

[1] Unexplained terms in the statement are defined in the main body of the text.

In [26] a general uniformization theorem valid for arbitrary Shimura varieties of PEL-type is proved. However, in this generality, one only obtains uniformization along the *basic locus* in the special fiber. As soon as this basic locus has dimension strictly smaller than that of the whole special fiber, the uniformizing formal scheme is no longer p-adic; only when all points of the special fiber are basic can there be p-adic uniformization. It is then a matter of experience that, in these very rare cases, the uniformizing formal scheme is always a product of Drinfeld halfspaces. This is predicted in [23], and is also supported by the classification of Kottwitz of *uniform pairs* (G, μ), cf. [15], §6.

In the paper we are dealing with p-divisible groups, say over an algebraically closed field k of characteristic p, equipped with some complex multiplication and with a compatible polarization, and their associated Dieudonné modules. The (rational) Dieudonné modules of these p-divisible groups inherit these additional structures. In this paper, we treat the theory of these Dieudonné modules with additional structure on the most elementary level, not on the group-theoretical level. This is in contrast to Kottwitz's approach where one first fixes a suitable algebraic group G over $\mathbb{Q}_p$ and then describes these (rational) Dieudonné modules as elements of $G(W_{\mathbb{Q}}(k))$. We refer to [7], ch. XI, §1 for the theory of *augmented group schemes* with values in the tannakian category of isocrystals over k which links these two approaches. We feel that Kottwitz's approach (or that of augmented group schemes) would have been unnatural in our context, although it could most probably be used as an alternative method to obtain our results. As an example, one can compare the proof of Lemma 5.4 using our method, and the proof of the same statement given in Remark 6.3 using Kottwitz's method. In particular, as G. Laumon pointed out to us, it is quite likely that there is a close connection between the local invariants defined in section 3 and the Kottwitz invariant of [14].

We finally summarize the contents of the various sections. In section 2 we introduce the stack of CM-triples of a fixed *generalized CM-type* of arbitrary rank. In section 3 we introduce the local invariants in $\{\pm 1\}$ of a CM-triple, one for each place, when the rank is even. An interesting question that arises in this context is when these local invariants satisfy the product formula, cf. Question 3.3. We then specialize to rank 2, and show in section 4 that fixing the local invariants gives a decomposition of the stack of *almost principal CM-triples* into stacks with good finiteness properties, and with generic fiber equal to a Shimura variety. In section 5 we consider the local situation, i.e., consider p-divisible groups instead of abelian varieties, and exhibit conditions on *local CM-triples* that guarantee that they are all isogenous to each other and supersingular. In section 6, the results of section 5 are then used in order to prove an integral p-adic uniformization theorem. In the final section we give a rigid-analytic uniformization theorem, which allows us to also treat level structures that are no longer maximal in p.

As pointed out above, there is a basic difference between the statements of Cherednik's theorem (1.1) and of Theorem 1.1. In Cherednik's theorem there are no hypotheses on the local extension $F_w/\mathbb{Q}_p$, whereas we have to assume that the local extension is trivial at p. However, it seems very likely that our uniformization theorem is valid also for non-trivial local extensions. This would require a generalization of our interpretation of the Drinfeld formal halfplane in [16]. Such a generalization is the subject of ongoing work with Th. Zink, comp. Remark 5.5.

The Cherednik-Drinfeld uniformization theorem was used in arithmetic applications, like level-raising, resp. level-lowering of modular forms and also in bounding the size of Selmer groups, cf. [25], esp. §4. It is to be hoped that similar applications can be found for our uniformization theorem.

We thank J. Tilouine for raising the question of the global consequences of our local theorem in [16]. This paper arose in trying to answer his query.

Notation.

For a number field F, $\Sigma(F)$ (resp. $\Sigma_f(F)$, resp. $\Sigma_\infty(F)$) denotes the set of places (resp. finite places, resp. infinite places). By $\bar{\mathbb{Q}}$ we denote the field of algebraic numbers in $\mathbb{C}$. For any p-adic field F, we denote by $\breve{F}$ the completion of the maximal unramified extension of F, and by $O_{\breve{F}}$ its ring of integers. For a perfect field k, we write $W(k)$ for its ring of Witt vectors, and $W(k)_{\mathbb{Q}}$ for its field of fractions.

2. Generalized CM-types

Let K be a CM-field, with totally real subfield F. We denote by $\bar{\mathbb{Q}}$ the subfield of $\mathbb{C}$ of algebraic numbers.

Definition 2.1. Let $n \geqslant 1$. *A generalized CM-type of rank n* for K is a function

$$r : \mathrm{Hom}_{\mathbb{Q}}(K, \bar{\mathbb{Q}}) \longrightarrow \mathbb{Z}_{\geqslant 0}, \qquad \varphi \mapsto r_\varphi,$$

such that

$$r_\varphi + r_{\bar{\varphi}} = n, \quad \forall \varphi \in \mathrm{Hom}_{\mathbb{Q}}(K, \bar{\mathbb{Q}}).$$

We note that the values of r are integers in the interval $[0, n]$. For $n = 1$, this notion reduces to the usual notion of a CM-type for K, i.e., a half-system of complex embeddings. The notion is equivalent to that of a *effective n-orientation* of K discussed in [12], V. A, p. 190.

A generalized CM-type r determines its *reflex field*, the subfield $E = E(r)$ of $\bar{\mathbb{Q}}$ characterized by

$$\mathrm{Gal}(\bar{\mathbb{Q}}/E) = \{\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \mid r_{\sigma\circ\varphi} = r_\varphi, \, \forall\varphi\}.$$

We will be interested in abelian varieties with action by O_K such that the O_K-action on the Lie algebra is given by a generalized CM-type.

Definition 2.2. Let r be a generalized CM-type of rank n for K with associated reflex field E. An abelian scheme A over an O_E-scheme S is *of CM-type r* if A is equipped with an O_K-action ι such that, for all $a \in O_K$,

$$\mathrm{char}(T, \iota(a)|\mathrm{Lie}\, A) = i\big(\prod_{\varphi\in\mathrm{Hom}(K,\bar{\mathbb{Q}})} (T - \varphi(a))^{r_\varphi}\big), \tag{2.1}$$

where $i : O_E \to \mathcal{O}_S$ is the structure homomorphism (*Kottwitz condition*).

It will sometimes be convenient to fix a CM-type $\Phi \subset \mathrm{Hom}_{\mathbb{Q}}(K, \bar{\mathbb{Q}})$ and to express the function r as a signature

$$((r_v, s_v))_{v\in\Sigma_\infty(F)}, \qquad r_v = r_\varphi, \; s_v = r_{\bar{\varphi}}, \; \varphi = \varphi_v \in \Phi. \tag{2.2}$$

Here $\varphi = \varphi_v$ induces the place $v \in \Sigma_\infty(F)$. In particular, we will sometime refer to (2.1) as the *signature condition.*

Let (A, ι) be an abelian scheme of CM-type r over an O_E-scheme S. We will consider polarizations $\lambda : A \to A^\vee$ such that the corresponding Rosati-involution induces the complex conjugation on K. Such triples (A, ι, λ) will be called *triples of CM-type* r, and K will be clear from the context.

Let $\mathcal{M}_r$ be the stack of triples of CM-type r. We note that $\mathcal{M}_r$ is a Deligne-Mumford stack locally of finite type over Spec O_E, where $E = E(r)$ is the reflex field of the generalized CM-type r. However, it is highly reducible. We will try to separate connected components by introducing additional invariants. In fact, we will be mostly interested in triples of the following kind.

Definition 2.3. A triple of CM-type r is *almost principal* if there exists a (possibly empty) finite set $\mathcal{N}_0$ of prime ideals $\mathfrak{q}$ of K, all lying above prime ideals of F which do not split in K/F, such that, setting $\mathfrak{n} = \prod_{\mathfrak{q}\in\mathcal{N}_0} \mathfrak{q}$, we have

$$\mathrm{Ker}\,\lambda \subset A[\iota(\mathfrak{n})].$$

3. Local invariants

Let r be a generalized CM-type of rank n for the CM-field K. In this section we assume that n is *even.*

We first suppose that $S = \operatorname{Spec} k$, where k is a field that is, at the same time, an O_E-algebra. To a triple (A, ι, λ) of CM-type r over S, we will attach a local invariant

$$\mathrm{inv}_v(A, \iota, \lambda)^\natural \in F_v^\times/\mathrm{Nm}(K_v^\times)$$

for every place v of F. In addition, we let

$$\mathrm{inv}_v(A, \iota, \lambda) = \chi_v(\mathrm{inv}_v(A, \iota, \lambda)^\natural) = \pm 1,$$

where χ_v is the character of $F_v^\times$ associated to K_v/F_v. In particular, if v is non-archimedean and split in K, then the invariant is trivial.

a) *Archimedean places.* If v is archimedean, we set

$$\mathrm{inv}_v(A, \iota, \lambda) = \mathrm{inv}_v(A, \iota, \lambda)^\natural = (-1)^{r_\varphi + n(n-1)/2}.$$

Here φ is either of the two complex embeddings of K_v. Recall that n is even. Note that the factor $(-1)^{n(n-1)/2}$ is included in analogy with the standard definition of the discriminant of a quadratic or hermitian form. There it is included so that the discriminant depends only on the Witt class of the form, i.e., does not change if a hyperbolic plane is added.

b) *Non-archimedean places not dividing* $\operatorname{char} k$. Let $v|\ell$. We fix a trivialization of the ℓ-power roots of unity in an algebraic closure $\bar{k}$ of k, i.e. an isomorphism $\mathbb{Z}_\ell(1)_{\bar{k}} \simeq \mathbb{Z}_{\ell\,\bar{k}}$.

Let $V_\ell(A)$ be the rational ℓ-adic Tate module of A. Then $V_\ell(A)$ is a free $K \otimes \mathbb{Q}_\ell$-module of rank n. Due to the trivialization of $\mathbb{Z}_\ell(1)_{\bar{k}}$, the polarisation λ determines an alternating bilinear form

$$\langle\, ,\rangle_\lambda \,: V_\ell(A) \times V_\ell(A) \longrightarrow \mathbb{Q}_\ell\,, \tag{3.1}$$

satisfying

$$\langle \iota(a)x, y\rangle_\lambda \,=\, \langle x, \iota(\bar{a})y\rangle_\lambda \quad , \quad a \in O_K. \tag{3.2}$$

According to the decomposition $F \otimes \mathbb{Q}_\ell = \prod_{w|\ell} F_w$, we can write

$$V_\ell(A) = \bigoplus\nolimits_{w|\ell} V_w(A)\,, \tag{3.3}$$

where $V_w(A)$ is a free K_w-module of rank n. We can write

$$\langle\, ,\rangle_\lambda \,= \sum\nolimits_{w|\ell} \mathrm{Tr}_{F_w/\mathbb{Q}_\ell} \langle\, ,\,\rangle_{\lambda,w}\,, \tag{3.4}$$

where $\langle\, ,\rangle_{\lambda,w}$ is an alternating F_w-bilinear form on $V_w(A)$. Now K_v is a field by assumption. Let $\delta_v \in K_v^\times$ with $\bar{\delta}_v = -\delta_v$. Then we can write

$$\langle x, y\rangle_{\lambda,v} \,= \mathrm{Tr}_{K_v/F_v}(\delta_v^{-1} \cdot (x, y)_v)\,, \tag{3.5}$$

for a unique K_v/F_v-hermitian form $(\cdot\, ,\cdot)_v$ on $V_v(A)$.

We then define the local invariant at v as the discriminant of the hermitian form,

$$\mathrm{inv}_v(A, \iota, \lambda)^\natural = \mathrm{disc}\,(\;,\;)_v \in F_v^\times/\mathrm{Nm}(K_v^\times). \tag{3.6}$$

Recall that

$$\mathrm{disc}\,(\;,\;)_v = (-1)^{n(n-1)/2}\,\det((x_i, x_j)_v) \in F_v^\times/\mathrm{Nm}(K_v^\times), \tag{3.7}$$

where $\{x_i\}$ is any K_v-basis for $V_v(A)$. This is well-defined, independent of the auxiliary choices made. Indeed, any two choices of δ_v differ by an element in $F_v^\times$ and a different choice changes the discriminant by a factor in $F_v^{\times, n} \subset \mathrm{Nm}(K_v^\times)$ (recall that n is even). Similarly, a different trivialization of $\mathbb{Z}_\ell(1)_{\bar{k}}$ leaves the discriminant unchanged. We also note that $\mathrm{inv}_v(A, \iota, \lambda)$ is unchanged after any base change $k \to k'$.

Note that $\mathrm{inv}_v(A, \iota, \lambda) = 1$ for almost all places. More precisely, suppose that $v|\ell$ where ℓ is an odd prime that is unramified in K and ℓ does not divide $|\mathrm{Ker}\,(\lambda)|$. Then

the Tate module $T_\ell(A) \subset V_\ell(A)$ is a free $O_K \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$-module of rank n and is self dual with respect to $\langle\ ,\ \rangle_\lambda$. Taking δ_v to be a unit, we obtain a unimodular hermitian lattice $T_\ell(A)_v$ in $V_v(A)$, and hence $\mathrm{disc}(\ ,\ )_v$ is a unit and hence a norm.

c) *Non-archimedean places dividing* $p = \mathrm{char}\, k$. Let $v|p$. Let $\bar{k}$ be an algebraic closure of k, and let $M = M(A_{\bar{k}})$ be the Dieudonné module of the abelian variety $A_{\bar{k}} = A \otimes_k \bar{k}$. Then $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$ is a free $K \otimes_{\mathbb{Z}} W(\bar{k})$-module of rank $2n$, with the σ-linear Frobenius operator $\underline{F}$. Under the decomposition $F \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{w|p} F_w$, we obtain a decomposition

$$M_{\mathbb{Q}} = \bigoplus\nolimits_{w|p} M_{\mathbb{Q},w}\,,$$

where $M_{\mathbb{Q},w}$ is a free $F_w \otimes_{\mathbb{Z}_p} W(\bar{k})$-module of rank $2n$, stable under the Frobenius. In particular, for our fixed v, $M_{\mathbb{Q},v}$ is a free $K_v \otimes_{\mathbb{Z}_p} W(\bar{k})$-module of rank n. As in the ℓ-adic case, $M_{\mathbb{Q},v}$ is equipped with a hermitian form

$$(\,,)_v : M_{\mathbb{Q},v} \times M_{\mathbb{Q},v} \longrightarrow K_v \otimes_{\mathbb{Z}_p} W(\bar{k})\,,$$

which, in this case, satisfies the additional condition

$$(\underline{F}x, \underline{F}y)_v = p \cdot (x,y)_v^\sigma \quad , \quad \forall x, y \in M_{\mathbb{Q},v}\,.$$

Let

$$N_{\mathbb{Q},v} = \bigwedge\nolimits^{n}_{K_v \otimes_{\mathbb{Z}_p} W(\bar{k})} M_{\mathbb{Q},v}\,.$$

Then $N_{\mathbb{Q},v}$ is a free $K_v \otimes_{\mathbb{Z}_p} W(\bar{k})$-module of rank 1, equipped with a hermitian form

$$(\,,)_v : N_{\mathbb{Q},v} \times N_{\mathbb{Q},v} \longrightarrow K_v \otimes_{\mathbb{Z}_p} W(\bar{k})$$

satisfying

$$(\underline{F}x, \underline{F}y)_v = p^n \cdot (x,y)_v^\sigma\,.$$

Furthermore, $N_{\mathbb{Q},v}$ is an isoclinic rational Dieudonné module of slope $\frac{n}{2}$. Since n is even, there exists $x_0 \in N_{\mathbb{Q},v}$ with $\underline{F}x_0 = p^{\frac{n}{2}} \cdot x_0$, such that x_0 generates the $K_v \otimes_{\mathbb{Z}_p} W(\bar{k})$-module $N_{\mathbb{Q},v}$. From

$$p^n \cdot (x_0, x_0)_v = (p^{\frac{n}{2}} x_0, p^{\frac{n}{2}} x_0)_v = (\underline{F}x_0, \underline{F}x_0)_v = p^n \cdot (x_0,x_0)_v^\sigma\,,$$

it follows that $(x_0,x_0)_v \in F_v^\times$. The local invariant is the residue class

$$\mathrm{inv}_v(A, \iota, \lambda)^\natural = (-1)^{n(n-1)/2}\,(x_0,x_0)_v \in F_v^\times/\mathrm{Nm}(K_v^\times).$$

It is easy to see that this definition is independent of all choices, i.e., of the algebraic closure $\bar{k}$ of k, of the scaling δ_v of the hermitian form $(\,,)_v$, and of the choice of the generator x_0 of $N_{\mathbb{Q},v}$ above.

Remark 3.1. The invariant at a p-adic place is analogous to Ogus's *crystalline discriminant*, cf. [19].

Proposition 3.2. *Let $(A, \iota, \lambda) \in \mathcal{M}_r(S)$, where S is a connected scheme of finite type over a field. Then for every place v of F, the function*

$$s \longmapsto \mathrm{inv}_v(A_s, \iota_s, \lambda_s)$$

is constant on S.

Proof. The assertion is trivial for archimedean places and for places split in K. It is obvious for places over ℓ invertible in $\mathcal{O}_S$ because then the ℓ-adic Tate module $V_\ell(A_s)$ is the fiber of a lisse ℓ-adic sheaf on S. Now suppose that $p \cdot \mathcal{O}_S = 0$. Then we may assume that S is a scheme of finite type over an algebraically closed field k, and then that S is a smooth affine curve. But then we may choose a lifting (T, F_T) of (S, F_S) over $W(k)$ and consider the value H of the crystal of A on the PD-embedding $S \hookrightarrow T$. Then the Dieudonné module of A_s is equal to the fiber at s of H, and the invariant $\mathrm{inv}_v(A_s, \iota_s, \lambda_s)$

depends on the value at s of a section of a lisse $\mathbb{Z}_p$-sheaf on S, defined by the analogous procedure as above, replacing $W(k)$ by $\mathcal{O}_T$. Hence it is locally constant. $\square$

Question 3.3. Let k be any field and consider a CM-triple $(A, \iota, \lambda) \in \mathcal{M}_r(k)$. When is the product formula satisfied,

$$\prod_v \mathrm{inv}_v(A, \iota, \lambda) = 1?$$

Note that one can suppose in this question that k is algebraically closed. This question looses its sense when the generalized CM-type cannot be read off from (A, ι, λ). This happens for instance when $F = \mathbb{Q}_p$ and $\mathrm{char}\, k = p$, where p ramifies in K, cf. [21]. Indeed, in this case the two $\mathbb{Q}_p$-embeddings of K into $\bar{\mathbb{Q}}_p$ induce identical homomorphisms from O_K into k, and hence the generalised CM-type for (A, ι, λ) can be changed arbitrarily, without violating the Kottwitz condition (2.1).

Proposition 3.4. *The product formula is satisfied in the following cases.*
(i) If $\mathrm{char}\, k = 0$.
(ii) If $\mathrm{char}\, k = p > 0$, *and* (A, ι, λ) *can be lifted as an object of* $\mathcal{M}_r$ *to a DVR with residue field* k *and fraction field of characteristic zero. More generally, the same is true if there exists an* O_K*-linear isogeny* $\alpha : A' \to A$, *such that* A' *can be lifted to characteristic zero in the previous sense, compatible with the isogeny action* $K \to \mathrm{End}(A') \otimes \mathbb{Q}$ *and the polarization* $\alpha^*(\lambda)$ *of* A', *provided that* A' *is of generalized CM-type* r' *with* $r'_\varphi \equiv r_\varphi \bmod 2, \forall \varphi$.

Proof. The condition on the generalised CM-type in (ii) makes sense since in characteristic zero K acts on the Lie algebra, and the CM-type of the Lie algebra can be read off from this action of K.

Let us first prove (i). We may assume first that k is a field extension of finite type of $\mathbb{Q}$, and then, by the invariance of our definitions under extension of scalars, that $k = \mathbb{C}$. Then the first rational homology group $U = H_1(A, \mathbb{Q})$ is equipped with an action of K and a symplectic form $\langle\, ,\, \rangle$ which, after extension of scalars to $\mathbb{Q}_\ell$, gives the ℓ-adic Tate module of A. Fix $\delta \in K$ and its associated standard CM-type Φ as in the beginning of the next section. By the same procedure as above, U is equipped with a hermitian form $(\, ,\,)_U$, comp. also (4.5) below. Let $\mathrm{inv}_v(U)$ be the local invariants of this hermitian vector space, comp. loc. cit. They satisfy the product formula. By the compatibility with the ℓ-adic Tate modules, it follows for any $v \in \Sigma_f(F)$ that $\mathrm{inv}_v(U) = \mathrm{inv}_v(A, \iota, \lambda)$. To complete the proof, we have to see that, for any archimedean place v, we have $\mathrm{inv}_v(U) = (-1)^{r_\varphi + n(n-1)/2}$, i.e., that $\mathrm{sig}((U,\ (\ ,\ )_U) = ((r_\varphi, r_{\bar\varphi}))_{\varphi\in\Phi}$. This is proved in the next section, right after (4.6).

Now let us prove (ii). Let $(\tilde{A}, \tilde{\iota}, \tilde{\lambda})$ be a CM-triple of type r over a DVR O with residue field k and fraction field L of characteristic 0 lifting (A, ι, λ). Since the ℓ-adic Tate modules of A and $\tilde{A}_L := \tilde{A} \otimes_O L$ can be canonically identified, it follows that $\mathrm{inv}_v(A, \iota, \lambda) = \mathrm{inv}_v\big((\tilde{A}, \tilde{\iota}, \tilde{\lambda})_L\big)$ for any finite v not lying over p.

By p-adic Hodge theory, there is a canonical isomorphism

$$V_p(\tilde{A}_L) \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}} \simeq M(A)_{\mathbb{Q}} \otimes_{W(k)_{\mathbb{Q}}} B_{\mathrm{crys}}\,, \tag{3.8}$$

compatible with all structures on both sides, in particular, with the Frobenii, with the K-actions on both sides and with the polarization forms, cf. [10, 27]. Here B_{crys} denotes Fontaine's period ring, cf. [11]. Moreover, after extension of scalars under the inclusion $B_{\mathrm{crys}} \subset B_{\mathrm{dR}}$, this isomorphism is compatible with the filtrations on both sides.

Decomposing both sides with respect to the actions of $F \otimes \mathbb{Q}_p$, we obtain, for any place $v|p$ of F that does not split in K, corresponding isomorphisms of K_v-modules

$$V_v(\tilde{A}_L) \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}} \simeq M(A)_{\mathbb{Q},v} \otimes_{W(k)_{\mathbb{Q}}} B_{\mathrm{crys}}\,,$$

where $V_v(\tilde{A}_L)$ is the summand of $V_p(\tilde{A}_L)$ corresponding to v in the product decomposition $F \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{w|p} F_w$, and where the other notation is taken from the definition of the local invariant at v, given in c) above.

Let $S_v = \bigwedge_{K_v}^n V_v(\tilde{A}_L)$. Then we obtain an isomorphism between free $K_v \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}}$-modules of rank one,

$$S_v \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}} \simeq N_{\mathbb{Q},v} \otimes_{W(k)_{\mathbb{Q}}} B_{\mathrm{crys}} . \tag{3.9}$$

Now we saw in the course of the definition of $\mathrm{inv}_v(A, \iota, \lambda)$ above, that $N_{\mathbb{Q},v} = N_{0,v}(\frac{n}{2})$, where $N_{0,v}$ is a multiple of the unit object in the category of filtered Dieudonné modules (even as a *filtered* Dieudonné module). Untwisting and taking on both sides of (3.9) the subsets in the 0-th filtration part where the Frobenius acts trivially, we obtain an isomorphism of Galois modules

$$S_v \simeq \bar{N}_{0,v}(\frac{n}{2}) ,$$

where $\bar{N}_{0,v}$ is the Galois representation corresponding to $N_{0,v}$, cf. [11]. By the functoriality of this isomorphism, it is compatible with the hermitian forms on these one-dimensional K_v-vector spaces. It therefore follows from the definition of the local invariant, that

$$\mathrm{inv}_v(A, \iota, \lambda) = \mathrm{inv}_v\big((\tilde{A}, \tilde{\iota}, \tilde{\lambda})_L\big) . \tag{3.10}$$

Now the product formula for $(\tilde{A}, \tilde{\iota}, \tilde{\lambda})_L$, valid by (i), implies the assertion.

The addendum in (ii) follows easily by observing that the local invariants only depend on the isogeny class of the CM-triple in the sense made precise in the statement. $\square$

Remark 3.5. The proof above is analogous to the proof in [14] that the *Kottwitz invariant* is trivial.

4. Formulation of the moduli problem

In this section we fix a CM-field K with totally real subfield F and an element $\delta \in K^\times$ with $\bar{\delta} = -\delta$. This element determines a (standard) CM type Φ by the condition

$$\Phi = \{ \ \varphi \in \mathrm{Hom}_{\mathbb{Q}}(K, \bar{\mathbb{Q}}) \mid \mathrm{Im}(\varphi(\delta)) > 0 \ \}.$$

Let r be a generalized CM-type of rank 2 with reflex field $E = E(r)$, and let $\big(V, (\ , \ )_V\big)$ be a hermitian vector space of dimension 2 over K such that $V \otimes_{K,\varphi} \mathbb{C}$ has signature $(r_\varphi, r_{\bar{\varphi}})$, for every $\varphi \in \Phi$. Note that this is consistent with (2.2).

We also fix a function h on the set of prime ideals of F,

$$h : \Sigma_f(F) \longrightarrow \{0, 1, 2\}, \quad \mathbf{p} \mapsto h_{\mathbf{p}} , \tag{4.1}$$

with finite support contained in the set of prime ideals that are non-split in K/F. We also sometimes write $h_{\mathbf{q}}$ for $h_{\mathbf{p}}$, where $\mathbf{q}$ denotes the prime ideal of K over $\mathbf{p}$, and we let

$$\mathfrak{n} = \mathfrak{n}(h) = \prod_{\substack{\mathbf{q} \\ h_{\mathbf{q}} \neq 0}} \mathbf{q}.$$

Definition 4.1. Given data $(K/F, r, h, V)$, let $\mathcal{M}_{r,h,V}$ be the DM-stack over (Sch/O_E) with

$$\mathcal{M}_{r,h,V}(S) = \text{category of triples } \ (A, \iota, \lambda) \ \text{ of CM-type } r \text{ over } S,$$

satisfying the following two conditions.

(i) The triple (A, ι, λ) of CM-type r is almost principal with $\mathrm{Ker}\, \lambda \subset A[\iota(\mathfrak{n})]$ and

$$|\mathrm{Ker}\ \lambda| = \prod\nolimits_{\mathbf{q}} N(\mathbf{q})^{h_{\mathbf{q}}} . \tag{4.2}$$

Here $N(\mathbf{q}) = |O_K/\mathbf{q}|$.

(ii)

$$\mathrm{inv}_v(A, \iota, \lambda) = \mathrm{inv}_v(V) , \quad \forall v \in \Sigma(F) . \tag{4.3}$$

Note that the stack $\mathcal{M}_{r,h,V}$ may be non-flat at certain places ν of E. Let ν lie above a prime number p. Then non-flatness can occur if p ramifies in K, or if p is divided by prime ideals of F in the support of h.

Let G be the group of unitary similitudes of V with similitude factor in $\mathbb{Q}$, i.e., the linear algebraic group over $\mathbb{Q}$, with values in a $\mathbb{Q}$-algebra R given by

$$G(R) = \{g \in \mathrm{GL}_{K\otimes R}(V \otimes_{\mathbb{Q}} R) \mid (gv, gw) = c(g)\cdot(v,w),\ c(g)\in R^\times\}.$$

In particular, if we let

$$\Phi_a = \{\ \varphi\in\Phi \mid r_\varphi = a\ \},$$

then[2]

$$G(\mathbb{R}) \simeq \Big(\prod_{\varphi\in\Phi_1}\mathrm{GU}(1,1)\times\prod_{\varphi\in\Phi_2\cup\Phi_0}\mathrm{GU}(2)\Big)_0$$

where the subscript 0 denotes the subgroup for which all the scale factors coincide. For $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$, let

$$\mathbf{h} : \mathbb{S} \longrightarrow G_{\mathbb{R}}, \qquad \mathbf{h} : z \longmapsto (\mathbf{h}_\varphi(z))_{\varphi\in\Phi}, \tag{4.4}$$

where

$$\mathbf{h}_\varphi(z) = \begin{cases}\begin{pmatrix} z & \\ & \bar z\end{pmatrix} & \text{for } \varphi\in\Phi_1,\\ z\cdot 1_2 & \text{for } \varphi\in\Phi_2,\\ \bar z\cdot 1_2 & \text{for } \varphi\in\Phi_0.\end{cases}$$

Note that $c\circ\mathbf{h}(z) = |z|^2$. Also note that the image of the scale map is

$$c(G(\mathbb{R})) = \begin{cases}\mathbb{R}^\times & \text{if } \Phi_2\cup\Phi_0 \text{ is empty, and}\\ \mathbb{R}^\times_{>0} & \text{otherwise.}\end{cases}$$

We let $G(\mathbb{R})^+$ be the subgroup of $G(\mathbb{R})$ for which the scale is positive.

Proposition 4.2. *Assume that all finite places v of F with $v|2$ are unramified in K/F.*
(i) The set $\mathcal{M}_{r,h,V}(\mathbb{C})$ of complex points of the moduli space $\mathcal{M}_{r,h,V}$ is non-empty if and only if the following compatibility conditions between the function h and the invariants of V are satisfied.

(1) *If $h_{\mathbf{p}_v} = 0$ and v is inert in K/F, then $\mathrm{inv}_v(V) = 1$.*
(2) *If $h_{\mathbf{p}_v} = 2$, then $\mathrm{inv}_v(V) = 1$.*
(3) *If $h_{\mathbf{p}_v} = 1$, then v is inert in K/F and $\mathrm{inv}_v(V) = -1$.*

(ii) In this case $\mathcal{M}_{r,h,V}\otimes_{O_E}\mathbb{C}$ is the Shimura variety $\mathrm{Sh}_C(G,\mathbf{X})$ associated to the pair $(G,\mathbf{X})$, where $\mathbf{X}$ is the $G(\mathbb{R})$-conjugacy class of homomorphisms $\mathbf{h}:\mathbb{S}\to G_{\mathbb{R}}$ given by (4.4), and where the compact open subgroup C of $G(\mathbb{A}^\infty)$ is the stabilizer of an O_K-lattice in V satisfying conditions (4.8) and (4.9) in Lemma 4.3 below.

Proof. Let (A,ι,λ) be a CM-triple of type r over $\mathbb{C}$. Then $U = H_1(A,\mathbb{Q})$ is a 2-dimensional K-vector space, and the Riemann form determined by λ is an alternating, $\mathbb{Q}$-bilinear form $\langle\ ,\ \rangle : U\times U\to\mathbb{Q}$ such that $\langle\iota(a)x,y\rangle = \langle x,\iota(\bar a)y\rangle$, for all $a\in K$. There is then a unique K-valued hermitian form $(\ ,\ )_U$ on U such that

$$\langle x,y\rangle \ =\ \mathrm{tr}_{K/\mathbb{Q}}(\delta^{-1}(x,y)_U), \tag{4.5}$$

where $\delta = -\bar\delta\in K^\times$ is the element fixed at the beginning of this section. For each place v of F, the hermitian space $(U,\ (\ ,\ )_U)$ has invariant

$$\mathrm{inv}_v(U,\ (\ ,\ )_U) = \chi_v(-\det((u_i,u_j)_U))\in\{\pm1\},$$

[2] We take the hermitian form on $\mathbb{C}^2$ to be $\mathrm{diag}(1,-1)$ for the Φ_1 factors and $\pm 1_2$ for the other factors.

where $\{u_1, u_2\}$ is a K-basis for U and $\chi_v(x) = (\delta^2, x)_v$ is the quadratic character associated to K_v/F_v. Here $(a, b)_v$ is the quadratic Hilbert symbol for F_v. By the compatibility of the $\mathbb{Q}_\ell$-bilinear extension of the Riemann form and the form (3.1) arising from the Weil pairing and by (4.3), we have

$$\mathrm{inv}_v(U,\ (\ ,\ )_U) = \mathrm{inv}_v(A, \iota, \lambda) = \mathrm{inv}_v(V,\ (\ ,\ )_V), \tag{4.6}$$

for all finite places v. On the other hand, under the isomorphism

$$\mathrm{Lie}\,(A) \ \overset{\sim}{\longleftarrow}\ U \otimes_{\mathbb{Q}} \mathbb{R} \ \overset{\sim}{\longrightarrow} \prod_{v \in \Sigma_\infty(F)} U \otimes_{F,v} \mathbb{R}, \tag{4.7}$$

the complex structure on $\mathrm{Lie}\,(A)$ induces a complex structure J on $U \otimes_{\mathbb{Q}} \mathbb{R}$ which preserves each of the factors on the right. On the v-th factor on the right side of (4.7), there is a complex structure

$$J_\delta = \iota(\delta) \otimes N_{K/F}(\delta)_v^{-\frac{1}{2}}.$$

Recall that $N_{K/F}(\delta)$ is totally positive. The signature condition (2.2) implies that, for $\varphi \in \Phi$, $J_\delta = J$ when $r_\varphi = 2$, $J_\delta = -J$ when $r_\varphi = 0$, and J_δ has eigenvalues $\pm i$ on $(U \otimes_{F,v} \mathbb{R}, J)$ when $r_\varphi = 1$. From this it follows that the signature

$$\mathrm{sig}(U,\ (\ ,\ )_U) = ((r_\varphi, r_{\bar\varphi}))_{\varphi \in \Phi}$$

coincides with that of V. Together with (4.6), this implies that there is an isometry $\eta : U \overset{\sim}{\longrightarrow} V$ of hermitian spaces over K. Via η and (4.7), the action of $\mathbb{C}^\times$ on $\mathrm{Lie}\,(A)$ yields a homomorphism $\mathbf{h}_A : \mathbb{S} \longrightarrow G_{\mathbb{R}}$. It is conjugate by $G(\mathbb{R})^+$ to the homomorphism $\mathbf{h}$ defined by (4.4).

Next, we must take into account the almost principal condition (i) in Definition 4.1, which amounts to the following. For convenience, we write $(\ ,\ )_V$ of the hermitian form on V.

Lemma 4.3. *(i) Let $L = H_1(A, \mathbb{Z}) \subset U$ with dual lattice $L^\vee$ with respect to $\langle\ ,\ \rangle$. Let $M = \eta(L)$, resp. $M^\vee = \eta(L^\vee)$, be the image of L, resp. $L^\vee$, under η. Then M and $M^\vee$ are O_K-lattices in V satisfying the conditions*

$$M \subset M^\vee \subset \mathfrak{n}^{-1} M, \tag{4.8}$$

and

$$|M^\vee/M| = \prod\nolimits_{\mathbf{q}} N(\mathbf{q})^{h_{\mathbf{q}}}. \tag{4.9}$$

Furthermore,

$$M^\vee = \{\ x \in V \mid \ \delta^{-1}(x, M)_V \subset \partial_K^{-1}\ \},$$

with ∂_K the different of $K/\mathbb{Q}$.

The subgroup

$$G(\mathbb{A}^\infty)^0 = \{\ g \in G(\mathbb{A}^\infty) \mid c(g) \in \widehat{\mathbb{Z}}^\times\ \}$$

of $G(\mathbb{A}^\infty)$ acts on the set of such lattices.

(ii) Assume that all finite places v of F with $v|2$ are unramified in K/F. Then, for any finite place v of F, a lattice in V_v satisfying the local analogues of (4.8) and (4.9) is unique up to isometry. In particular, the isometry group $U(V)(F_v)$ acts transitively on the set of such lattices.

Proof. For a finite place $v \in \Sigma_f(F)$ of F, choose $\delta_v \in K_v^\times$ such that $\bar\delta_v = -\delta_v$ and $\delta_v O_{K_v} = \partial_{K,v}$. Note that, for almost all places, we can take $\delta_v = \delta$. Then the lattice M_v in V_v has dual lattice

$$(M^\vee)_v = \{\ x \in V \mid \ \delta_v \delta^{-1}(x, M_v)_V \subset O_{K,v}\ \},$$

with respect to the hermitian form $h_v := \delta_v\delta^{-1}(\ ,\ )_V$. Note that the 2-dimensional hermitian spaces (V_v, h_v) and $(V_v, (\ ,\ )_V)$ are related to each other by scaling and hence are isometric. Now apply the local theory of hermitian lattices.

If $h_{\mathbf{p}_v} = 0$, then $\mathfrak{n}_v = O_{K,v}$ so that $M_v^\vee = M_v$, and M_v is unimodular with respect to h_v. If v is split in K/F, there is a unique 2-dimensional hermitian space V_v, and a self dual lattice M_v in it is unique up to isometry. If v is inert in K/F, then the space (V_v, h_v) is split and the lattice M_v is unique up to isometry, [13], Theorem 7.1. If v is ramified in K/F (and hence non-dyadic by our assumption), the isometry class of M_v is determined by $\det(M_v) \in F_v^\times/\mathrm{Nm}(K_v^\times)$, [13], Proposition 8.1 (a). But since the class of V_v is already fixed, it follows that there is a unique isometry class of self-dual M_v's in V_v.

Next suppose that $h_{\mathbf{p}_v} = 2$, so that $M_v^\vee = \pi_{K_v}^{-1} M_v$ and M_v is π_{K_v}-modular. Note that, by our assumption about the support of the function h, the case where v is split in K/F is excluded. If v is inert in K/F, then the lattice M_v is unique up to isometry, [13], Theorem 7.1, and the space V_v is split. If v is ramified in K/F, then the existence of a π_{K_v}-modular lattice implies that V_v is split and, again, the lattice M_v is unique up to isometry, [13], Proposition 8.1 (b).

Finally, suppose that $h_{\mathbf{p}_v} = 1$. In particular, M_v is not modular and has a Jordan decomposition of type $(1) \oplus (\pi_{K_v})$. If v is inert in K/F, it follows that V_v is anisotropic and that M_v is unique up to isometry, [13], Theorem 7.1. On the other hand, if v is ramified in K/F, and hence non-dyadic, there are no π_{K_v}-modular lattices of rank 1, [13], Proposition 8.1. so that the condition $h_{\mathbf{p}_v} = 1$ cannot occur for ramified places v. □

The previous lemma and its proof imply that the conditions (1)–(3) in Proposition 4.2, (i), are equivalent to the existence of an O_K-lattice M in V satisfying (4.8) and (4.9). We fix such a lattice and let $C \subset G(\mathbb{A}^\infty)$ be its stabilizer. Note that, by the lemma, $c(C) = c(G(\mathbb{A}^\infty)^0)$ and the set of all lattices in V satisfying (4.8) and (4.9) can be identified with $G(\mathbb{A}^\infty)^0/C$. Let $\mathbf{X}^+$ be the $G(\mathbb{R})^+$ conjugacy class of $\mathbf{h}$. The pair $(\mathbf{h}_A, M) \in \mathbf{X}^+ \times G(\mathbb{A}^\infty)^0/C$ depends on the choice of isometry η. Removing this dependence, we have a map

$$\mathcal{M}_{r,h,V}(\mathbb{C}) \longrightarrow U(V)(\mathbb{Q})\backslash \mathbf{X}^+ \times G(\mathbb{A}^\infty)^0/C \xrightarrow{\sim} G(\mathbb{Q})\backslash \mathbf{X} \times G(\mathbb{A}^\infty)/C. \tag{4.10}$$

Here note that $c(G(\mathbb{A}^\infty)^0) = \widehat{\mathbb{Z}}^\times$ and that $c(G(\mathbb{Q}) \cap G(\mathbb{A}^\infty)^0) = 1$ if $c(G(\mathbb{R})) = \mathbb{R}_+^\times$ and ± 1 if $c(G(\mathbb{R})) = \mathbb{R}^\times$. It is easily checked that (4.10) is surjective and induces a bijection on the set of isomorphism classes on the left hand side. □

Remark 4.4. $\mathcal{M}_{r,h,V} \otimes_{O_E} E$ is the canonical model in the sense of Deligne of the Shimura variety $\mathrm{Sh}_C(G, \mathbf{X})$, but we will not stop to show this here.

Remark 4.5. We will also use the following variant of $\mathcal{M}_{r,h,V}$. We fix a prime number p and an O_K-lattice M in V satisfying (4.8) and (4.9), and let $C_M^p \subset G(\mathbb{A}^{\infty,p})$ be the stabilizer of $M \otimes \widehat{\mathbb{Z}}^p$ in $G(\mathbb{A}^{\infty,p})$. Let $C^p \subset G(\mathbb{A}^{\infty,p})$ be an open compact subgroup which is a subgroup of finite index in C_M^p. Let $O_{E_{(p)}}$ be the localization of O_E at p. The variant $\mathcal{M}_{r,h,V}(C^p)$ is the stack over $(Sch/O_{E_{(p)}})$ which, in addition to (A, ι, λ) satisfying conditions (4.2) and (4.3), fixes a level structure mod C^p, i.e. an isomorphism compatible with ι and with the alternating forms on both sides up to a unit in $\widehat{\mathbb{Z}}^p$,

$$\widehat{T}^p(A) \simeq M \otimes \widehat{\mathbb{Z}}^p \bmod C^p \,,$$

in the sense of Kottwitz [14]. Here on the RHS, we use the alternating form

$$\langle x, y\rangle_V = \mathrm{tr}_{K/\mathbb{Q}}(\delta^{-1}(x,y)_V)\,. \tag{4.11}$$

Note that, due to the existence of the level structure, we only have to require the condition (4.3) for the places v over p — for all other places it is automatic. If $C^p = C_M^p$, then by Lemma 4.3, $\mathcal{M}_{r,h,V}(C^p) = \mathcal{M}_{r,h,V} \otimes_{O_E} O_{E_{(p)}}$.

5. Uniformizing primes

In this section we consider the local situation. We fix a prime number p and an algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Let F be a finite extension of $\mathbb{Q}_p$ with $|F:\mathbb{Q}_p| = d$, and let K/F be an étale algebra of rank 2. We begin with the obvious local analogues of the definitions of section 2.

A generalized CM-type r of rank n relative to K/F is a function

$$r : \mathrm{Hom}_{\mathbb{Q}_p}(K, \bar{\mathbb{Q}}_p) \longrightarrow \mathbb{Z}_{\geqslant 0}, \qquad \varphi \mapsto r_\varphi,$$

such that $r_\varphi + r_{\bar\varphi} = n$ for all φ. Here $\bar\varphi(a) = \varphi(\bar a)$ where $a \mapsto \bar a$ is the non-trivial automorphism of K over F. The corresponding reflex field $E = E(r)$ is the subfield of $\bar{\mathbb{Q}}_p$ fixed by

$$\mathrm{Gal}(\bar{\mathbb{Q}}_p/E) := \{\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \mid r_{\tau\varphi} = r_\varphi,\ \forall\varphi\}.$$

Let O_E be the ring of integers of E and let π_E be a uniformizer of E.

Definition 5.1. A *triple of CM-type* r over an O_E-scheme S is a triple (X, ι, λ), where X is a p-divisible group over S of height $2nd$ and dimension nd, $\iota : O_K \to \mathrm{End}(X)$ is an action of the ring of integers of K on X satisfying the Kottwitz condition relative to r, and $\lambda : X \to X^\vee$ is a quasi-polarization with Rosati involution inducing the non-trivial automorphism on K/F.

Definition 5.2. Such a triple is called *almost principal* if either $K = F \oplus F$ and λ is principal, or K is a field and $\mathrm{Ker}\,\lambda$ is contained in $X[\iota(\pi)]$, where $\pi = \pi_K$ denotes a uniformizer of K.

In particular, when K is a field, $\mathrm{Ker}\,\lambda$ is a module over $O_K/\pi O_K$. We write the height of $\mathrm{Ker}\,\lambda$ in the form fh, where $f = [K^t : \mathbb{Q}_p]$ is the degree of the maximal unramified subfield K^t of K, so that $0 \le h \le n$. If $K = F \oplus F$, then $h = 0$.

Now let n be even.

Suppose that k is an algebraically closed field of characteristic p that is an O_E-algebra. Then, for a CM-triple (X, ι, λ) of type r over k, the construction of c) of section 3, applied to the Dieudonné module of (X, ι, λ), yields an invariant

$$\mathrm{inv}_v(X, \iota, \lambda)^\natural \in F^\times/\mathrm{Nm}(K^\times),$$

and a sign

$$\varepsilon = \mathrm{inv}_v(X, \iota, \lambda) = \chi(\mathrm{inv}_v(X, \iota, \lambda)^\natural) = \pm 1,$$

where χ is the quadratic character attached to K/F.

For the rest of this section, we assume that $n = 2$. For our description of p-adic uniformization, it will be important to know when an almost principal CM-triple (X, ι, λ) over k with given invariants h and ε is unique up to isogeny.

Definition 5.3. (i) We call $(K/F, r, h, \varepsilon)$ *uniformizing data of the first kind*, if $F = \mathbb{Q}_p$, K is a field, $r_\varphi = r_{\bar\varphi} = 1$,

$$h = \begin{cases} 0, & \text{if} \quad K/F \text{ is ramified} \\ 1, & \text{if} \quad K/F \text{ is unramified,} \end{cases}$$

and $\varepsilon = -1$.

(ii) We call $(K/F, r, h, \varepsilon)$ *uniformizing data of the second kind*, if K/F is an unramified field extension and r is of the following form: there exists a half-system Φ^t of elements of $\mathrm{Hom}_{\mathbb{Q}_p}(K^t, \bar{\mathbb{Q}}_p)$ such that

$$r_\varphi = \begin{cases} 0, & \text{if} \quad \varphi|K^t \in \Phi^t \\ 2, & \text{if} \quad \varphi|K^t \notin \Phi^t. \end{cases}$$

Note that $[K^t : F^t] = 2$. If $h = 0$, then $\varepsilon = 1$; if $h = 1$, then $\varepsilon = -1$.

(iii) We call $(K/F, r, h, \varepsilon)$ *uniformizing data of the third kind*, if $K = F \oplus F$ so $\varepsilon = 1$, $h = 0$, and

$$r_\varphi = \begin{cases} 0, & \text{if } \varphi \text{ factors through the first summand of } K = F \oplus F \\ 2, & \text{if } \varphi \text{ factors through the second summand of } K = F \oplus F. \end{cases}$$

Proposition 5.4. *Fix uniformizing data $(K/F, r, h, \varepsilon)$ of the first, second, or third kind. Let (X, ι, λ) and (X', ι', λ') be two almost principal CM-triples of type $(K/F, r, h, \varepsilon)$ over k. Then both X and X' are isoclinic p-divisible groups and there exists an O_K-linear isogeny $\alpha : X \to X'$ such that $\alpha^*(\lambda') = c\,\lambda$ with $c \in \mathbb{Z}_p^\times$.*

Proof. It suffices to prove that X and X' are isoclinic and to show the existence of the isogeny α with $\alpha^*(\lambda) = c\lambda'$ with $c \in \mathbb{Q}_p^\times$. That $c \in \mathbb{Z}_p^\times$ then follows from the fact that both CM-triples are almost principal with identical h.

We consider the three cases separately.

Suppose that (X, ι, λ) is of the first kind. Let $M = M(X)$ be its covariant Dieudonné module. Then M is a free $W(k)$-module of rank 4. Consider the slope decomposition of the corresponding rational Dieudonné module N. Each summand N_λ is stable under the action of K on N, hence has even dimension; we have $\lambda \geq 0$ if $N_\lambda \neq (0)$; furthermore, due to the polarization, $m_\lambda = m_{1-\lambda}$ for the multiplicities of the corresponding slope subspaces N_λ, resp. $N_{1-\lambda}$; finally, writing $\dim N_\lambda = m_\lambda d_\lambda$, where d_λ is the dimension of the simple isocrystal of slope λ, we have $4 = \sum_\lambda m_\lambda d_\lambda$. It follows that either $N = N_{\frac{1}{2}}$ and $m_{\frac{1}{2}} = 2$, or $N = N_0 \oplus N_1$, where both summands N_0 and N_1 have dimension 2 and $m_0 = m_1 = 2$. The first case means that X is isoclinic, the second that X is ordinary.

We need to exclude the ordinary case. Write $W(k)_\mathbb{Q} = \mathbb{Q} \otimes W(k)$. The isocrystal comes with its natural polarization pairing

$$\langle\, ,\, \rangle_0 : N \times N \longrightarrow W(k)_\mathbb{Q},$$

for which N_0 and N_1 are isotropic, and which corresponds to a non-degenerate pairing

$$[\, ,\,]_0 : N_0 \times N_1 \longrightarrow W(k)_\mathbb{Q}.$$

We choose bases e_0, e_1 for N_0, and f_0, f_1 for N_1 such that

$$\begin{aligned} \underline{V} e_i = e_i, \underline{F} e_i = p e_i;\ \underline{V} f_i = p f_i, \underline{F} f_i = f_i \text{ for } i = 0, 1; \\ [e_0, f_0]_0 = [e_1, f_1]_0 = 1,\ [e_0, f_1]_0 = [e_1, f_0]_0 = 0. \end{aligned} \tag{5.1}$$

In this case, $\mathrm{End}\,(N) = \mathrm{M}_2(\mathbb{Q}_p) \times \mathrm{M}_2(\mathbb{Q}_p)$. For $(b_0, b_1) \in \mathrm{M}_2(\mathbb{Q}_p) \times \mathrm{M}_2(\mathbb{Q}_p)$, we have

$$\langle (b_0, b_1)x, y\rangle_0 = \langle x, ({}^t b_1, {}^t b_0) y\rangle_0\,, \forall x, y \in N.$$

We may furthermore suppose that the action of K on N is given as follows. Let $K = \mathbb{Q}_p(\sqrt{\Delta})$. Putting $\delta = \sqrt{\Delta}$, the action of K on $N_0 \oplus N_1$ is given as

$$a + b\delta \mapsto \Big(\begin{pmatrix} a & b \\ \Delta b & a \end{pmatrix}, \begin{pmatrix} a & \Delta b \\ b & a \end{pmatrix} \Big). \tag{5.2}$$

The given polarization on X induces the pairing $\langle\, ,\, \rangle : N \times N \longrightarrow W(k)_\mathbb{Q}$. Comparing the involutions on K induced by the Rosati involutions of the two polarizations, we see that $\langle x, y\rangle = \langle \beta x, y\rangle_0$, where $\beta \in \mathrm{End}(N)$ anticommutes with K. Hence β is of the form $\beta = (\beta_0, \beta_1)$ with

$$\beta_0 = \begin{pmatrix} a & b \\ -\Delta b & -a \end{pmatrix},\ \beta_1 = \begin{pmatrix} a & -\Delta b \\ b & -a \end{pmatrix}. \tag{5.3}$$

Let us now calculate the local invariant according to the recipe of section 3. The hermitian form associated to the alternating form $\langle\, ,\, \rangle$ is given as

$$(x, y) = \frac{1}{2}(\langle \delta x, y\rangle + \delta\langle x, y\rangle) = \frac{1}{2}(\langle \beta\delta x, y\rangle_0 + \delta\langle \beta x, y\rangle_0). \tag{5.4}$$

Taking e_0, f_0 as $K \otimes_{\mathbb{Z}_p} W(k)$-basis of N, we obtain $x_0 = e_0 \wedge f_0$ as free generator of $\bigwedge^2 N$, with $\underline{F} x_0 = px_0$. Hence $\mathrm{inv}(X, \iota, \lambda)^\natural = -(x_0, x_0) \in \mathbb{Q}_p^\times / \mathrm{Nm}(K^\times)$. Now

$$(x_0, x_0) = \det \begin{pmatrix} (e_0, e_0) & (e_0, f_0) \\ (f_0, e_0) & (f_0, f_0) \end{pmatrix} = \det \begin{pmatrix} 0 & (e_0, f_0) \\ (f_0, e_0) & 0 \end{pmatrix}.$$

It follows that $-(x_0, x_0) \in \mathrm{Nm}_{K/\mathbb{Q}_p}(K^\times)$. This contradicts the imposed sign $\varepsilon = -1$ in the definition of uniformizing data of the first kind.

Applying the same reasoning to (X', ι', λ'), we obtain an isogeny $\alpha : X \longrightarrow X'$ compatible with the actions of K. However, $\lambda = \alpha^*(\lambda')\beta$ with $\beta \in \mathrm{End}(N)$ invariant under the Rosati involution of λ. By precomposing α with $\gamma \in \mathrm{End}_K(N)$, we change β into $\beta\gamma\gamma^*$. We consider the solutions of the equation $c\beta\gamma\gamma^* = 1$, with $c \in \mathbb{Q}_p^\times$ and $\gamma \in \mathrm{End}_K(N)$, as a torsor under the $\mathbb{Q}_p$-group of automorphisms of N which commute with $\underline{F}$ and preserve the polarization form up to a constant. Since this group has trivial first cohomology set, we may solve this equation, and may change α so that $\alpha^*(\lambda') = c\lambda$. This finishes the case of uniformizing data of the first kind.

If $(K/F, r, h)$ is a uniformizing data of the third kind, the assertion is proved in [26], Lemma 6.41. If $(K/F, r, h)$ is a uniformizing data of the second kind, then again the assertion is proved in [26], Lemma 6.41, provided that $h = 0$. In this case, the sign factor ε equals 1, since we are then calculating the discriminant of a hermitian vector space relative to an unramified quadratic extension which admits a self-dual lattice. An inspection of the proof of loc. cit. shows that the proof also applies to the case when $(K/F, r, h)$ is a uniformizing data of the second kind and $h = 1$, in which case $\varepsilon = -1$. □

Remark 5.5. As mentioned at the end of the introduction, it seems plausible that the notion of uniformizing data of the first kind can be generalized to include cases where F is a non-trivial extension of $\mathbb{Q}_p$. More precisely, fix an embedding $\varphi_0 : F \to \bar{\mathbb{Q}}_p$. Then the conditions on $(K/F, r, h, \varepsilon)$ become: K should be a field extension of F, r_φ should be equal to 1 if $\varphi|F = \varphi_0$, and should be equal to 0 or 2 otherwise. Finally, as before, h should be 0 or 1 depending on whether K/F is ramified or unramified, and ε should be -1.

6. Integral uniformization

In this section, we obtain integral p-adic uniformization under a whole set of assumptions that we now explain. We fix an embedding $\nu : \bar{\mathbb{Q}} \longrightarrow \bar{\mathbb{Q}}_p$. This embedding also determines a p-adic place ν of the reflex field E. We decompose $\mathrm{Hom}(K, \bar{\mathbb{Q}})$ into a disjoint sum according to the prime ideals $\mathbf{p}$ of F over p,

$$\mathrm{Hom}(K, \bar{\mathbb{Q}})_{\mathbf{p}} = \{\varphi \in \mathrm{Hom}(K, \bar{\mathbb{Q}}) \mid \nu \circ \varphi_{|F} \text{ induces } \mathbf{p}\}.$$

Then $\mathrm{Hom}(K, \bar{\mathbb{Q}})_{\mathbf{p}} = \mathrm{Hom}_{\mathbb{Q}_p}(K \otimes_F F_{\mathbf{p}}, \bar{\mathbb{Q}}_p)$. Let $r_{\mathbf{p}} = r|\mathrm{Hom}(K, \bar{\mathbb{Q}})_{\mathbf{p}}$ and $\varepsilon_{\mathbf{p}} = \mathrm{inv}_{\mathbf{p}}(V)$.

We make the assumption that $(K_{\mathbf{p}}/F_{\mathbf{p}}, r_{\mathbf{p}}, h_{\mathbf{p}}, \varepsilon_{\mathbf{p}})$ are uniformizing data of type $1, 2$, or 3, for all $\mathbf{p}|p$. We note that E_ν is the composite of the local reflex fields $E(r_{\mathbf{p}})$ (with $\mathbf{p}$ running over the prime ideals of F over p). Let κ_ν be the residue field of O_{E_ν}, and denote by $\bar{\kappa}_\nu$ its algebraic closure.

The proof of the integral uniformization theorem will be analogous to the proof of Theorem 6.30 in [26]. We will proceed according to the following steps. First we will show that all p-divisible groups (X, ι, λ) which arise from points (A, ι, λ) of $\mathcal{M}_{h,r,V}(\bar{\kappa}_\nu)$ are isogenous to each other, and are moreover *basic* in the sense of [15]. Then we prove that, in fact, all points (A, ι, λ) are isogenous to each other. This already yields an abstract integral p-adic uniformization theorem, as in [26]. In a third step, we make this abstract uniformization theorem explicit, by making use of the alternative moduli description of the Drinfeld halfplane in [16].

Let $(A_0, \iota_0, \lambda_0) \in \mathcal{M}_{h,r,V}(\bar{\kappa}_\nu)$. Let N be the isocrystal of A_0, with its action by $K \otimes \mathbb{Q}_p$ induced by ι_0, and its anti-symmetric polarization form induced by λ_0.

Lemma 6.1. *There is an isomorphism of $K \otimes W(\bar{\kappa}_\nu)$-modules*

$$N \simeq V \otimes W(\bar{\kappa}_\nu) \ ,$$

which respects the anti-symmetric bilinear forms on both sides.

Proof. We have an orthogonal decomposition with respect to the anti-symmetric form (4.11),

$$V \otimes \mathbb{Q}_p = \bigoplus\nolimits_{\mathbf{p}|p} V_{\mathbf{p}} \,. \tag{6.1}$$

There is a similar decomposition of N, orthogonal for the polarization form. Now $N_{\mathbf{p}}$ contains a parahoric lattice of type $h_{\mathbf{p}}$, i.e., a lattice Λ such that $\Lambda \subset \Lambda^\vee \subset \pi_{\mathbf{p}}^{-1}\Lambda$ where the dimension of $\Lambda^\vee/\Lambda$ over the residue field is equal to $h_{\mathbf{p}}$; this lattice is isomorphic to the extension of scalars of the parahoric lattice of type $h_{\mathbf{p}}$ in $V_{\mathbf{p}}$, cf [26], Theorem 3.16, comp. [26], 6.12 (we use the fact that $W(\bar{\kappa}_\nu)$ has no non-trivial étale coverings). Hence we get a fortiori the isomorphism of $K \otimes W(\bar{\kappa}_\nu)$-modules, as claimed.

Another way of obtaining this isomorphism is to note that for a complete discretely valued field with algebraically closed residue field, and a quadratic algebra over it, there is, up to isomorphism, exactly one hermitian space of given dimension. □

Using the isomorphism of Lemma 6.1, we can write the Frobenius operator on the left hand side as $b \otimes \sigma$, for a uniquely defined element $b \in G(W(\bar{\kappa}_\nu)_{\mathbb{Q}})$. We have $c(b) = p$, where $c : G \longrightarrow \mathbb{G}_m$ denotes the multiplier morphism. Recall Kottwitz's set $B(G)$ of σ-conjugacy classes of elements in $G(W_{\mathbb{Q}}(\bar{\kappa}_\nu))$, cf. [15].

Lemma 6.2. *The element $[b] \in B(G)$ is basic, and independent of $(A_0, \iota_0, \lambda_0)$.*

Proof. Corresponding to (6.1) there is an embedding of algebraic groups over $\mathbb{Q}_p$,

$$G_{\mathbb{Q}_p} \longrightarrow \prod\nolimits_{\mathbf{p}} G'_{\mathbf{p}} \,, \tag{6.2}$$

where the product runs over all primes of F over p, and where $G'_{\mathbf{p}}$ denotes the group of unitary similitudes of the hermitian space $V_{\mathbf{p}}$ (with similitude factor in $\mathbb{Q}_p$). Since the center of G is equal to the intersection of the center of $\prod_{\mathbf{p}} G'_{\mathbf{p}}$ with G, an element in $B(G)$ is basic if its image under the map $B(G) \to B(\prod_{\mathbf{p}} G'_{\mathbf{p}})$ is. Furthermore, using the long exact cohomology sequence associated to the injection (6.2), this last map is injective, since the map $\prod_{\mathbf{p}} G'_{\mathbf{p}}(\mathbb{Q}_p) \to (\prod_{\mathbf{p}} \mathbb{Q}_p^\times)/\mathbb{Q}_p^\times$ is surjective. Hence Proposition 5.4 implies that $[b]$ is basic and independent of $(A_0, \iota_0, \lambda_0)$. □

Remark 6.3. In the discussion above, we have adopted the point of view of Kottwitz, that is, we view N with its additional structure as given by an element in $\tilde{G}(W(\bar{\kappa}_\nu)_{\mathbb{Q}})$, where $\tilde{G}$ is a suitable reductive algebraic group over $\mathbb{Q}_p$. In fact, we have taken $\tilde{G}$ to be the localization at p of our group G over $\mathbb{Q}$. We follow here the method of Kottwitz for convenience only and to make our exposition more efficient, because we then can quote [26] . However, it should be pointed out that this point of view is not very natural in the framework of the present paper, and could be avoided. This can be done without any additional work if there is only one prime $\mathbf{p}$ over p.

A more sophisticated alternative proof of Lemma 6.2 uses the finite subset $B(G, \mu)$ of $B(G)$, for the conjugacy class of cocharacters μ associated to the conjugacy class of (4.4). Then $[b] \in B(G, \mu)$, by Mazur's inequality, cf. [22]. Using the bijections $B(G, \mu) \simeq B(G_{\mathrm{ad}}, \mu_{\mathrm{ad}}) \simeq \prod_{\mathbf{p}} B((G'_{\mathbf{p}})_{\mathrm{ad}}, \mu'_{\mathbf{p},\mathrm{ad}})$, cf. [15], 6.5., we obtain a bijection

$$B(G, \mu) \simeq \prod\nolimits_{\mathbf{p}} B(G'_{\mathbf{p}}, \mu'_{\mathbf{p}}),$$

where $\mu'_{\mathbf{p}}$ denotes the minuscule coweight of $G'_{\mathbf{p}}$ obtained from μ via (6.2). However, Proposition 5.4 implies that, for every $\mathbf{p}$, the image of $[b]$ in $B(G'_{\mathbf{p}}, \mu'_{\mathbf{p}})$ is the unique basic element in this set, which implies the assertion of Lemma 6.2.

Let $\mathbf{p}$ be of the first type. Then $(G'_{\mathbf{p}})_{\mathrm{ad}}$ is isomorphic to $(D^\times)_{\mathrm{ad}}$, where $D^\times$ denotes the algebraic group over $\mathbb{Q}_p$ associated to the quaternion division algebra over $\mathbb{Q}_p$. Furthermore, the coweight $\mu'_{\mathbf{p},\mathrm{ad}}$ of $(G'_{\mathbf{p}})_{\mathrm{ad}}$ is given by the unique nontrivial minuscule coweight. It then follows that $B(G'_{\mathbf{p}}, \mu'_{\mathbf{p}})$ consists only of the unique basic element in this set, cf. [15], §6. This gives a proof of Lemma 5.4 in the style of Kottwitz's view on isocrystals with additional structure and avoids the use of Proposition 5.4, cf. the remarks at the end of the Introduction. The primes of the second and the third type can also be viewed from this perspective, since for them $\mu'_{\mathbf{p},\mathrm{ad}}$ is central.

At this point we want to apply [26], Theorem 6.30. In the notation of that theorem, we want to show that $Z = Z'$. Let I be the linear algebraic group over $\mathbb{Q}$ of loc. cit., i.e.

$$I(\mathbb{Q}) = \{\alpha \in \mathrm{End}_K(A_0)^{0,\times} \mid \alpha^*(\lambda_0) = c\lambda_0,\, c \in \mathbb{Q}^\times\}$$

By loc. cit., I is an inner form of G.

Lemma 6.4. *The Hasse principle for I is satisfied.*

Proof. This follows from [14], §7. Indeed, we are here in the case A, for $n = 2$, and it is proved in loc. cit that the Hasse principle is satisfied in the case A, for any even n. $\square$

We may now apply [26], Theorem 6.30, and obtain an isomorphism

$$\Theta : I(\mathbb{Q})\backslash \mathcal{M} \times G(\mathbb{A}^{\infty,p})/C^p \simeq \mathcal{M}_{r,h,V}(C^p)^\wedge \ , \tag{6.3}$$

with notation as follows. On the RHS appears the formal completion of $\mathcal{M}_{r,h,V}(C^p)$ along its special fiber $\mathcal{M}_{r,h,V}(C^p) \otimes_{O_{E_{(p)}}} \kappa_\nu$. On the LHS, $\mathcal{M}$ denotes the formal moduli space $\breve{\mathcal{M}}$ over $O_{\breve{E}_\nu}$ with its Weil descent datum to O_{E_ν} associated in [26] to the data $(F \otimes \mathbb{Q}_p, K \otimes \mathbb{Q}_p, V \otimes \mathbb{Q}_p, b, r, \mathcal{L})$, attached to the situation at hand.

Let us describe more concretely the formal scheme $\breve{\mathcal{M}}$, with its Weil descent datum. We enumerate the prime ideals of F over p as follows:

$$\begin{array}{rl} \mathbf{p}_1, \dots, \mathbf{p}_r & \text{are uniformizing primes of the first kind} \\ \mathbf{p}_{r+1}, \dots, \mathbf{p}_{r+s} & \text{are uniformizing primes of the second kind} \\ \mathbf{p}_{r+s+1}, \dots, \mathbf{p}_{r+s+t} & \text{are uniformizing primes of the third kind .} \end{array}$$

The data b, μ decompose naturally under the embedding (6.2) as a product $b = \prod_i b_i$ and $\mu = \prod_i \mu_i$; here μ_i corresponds to the local generalized CM-type $r_{\mathbf{p}_i}$.

Let E_i denote the local Shimura field associated to $r_{\mathbf{p}_i}$. Let $\breve{\mathcal{M}}_i$ be the formal scheme over $\mathrm{Spf}\, O_{\breve{E}_i}$ whose values in a scheme $S \in \mathrm{Nilp}_{O_{\breve{E}_i}}$ are given by the set of isomorphism classes of quadruples $(X, \iota, \lambda, \varrho)$, where X is a p-divisible group over S with an action ι of $O_{K_{\mathbf{p}_i}}$ of CM-type $r_{\mathbf{p}_i}$ and a polarization λ with associated Rosati involution inducing the non-trivial automorphism of $K_{\mathbf{p}_i}$ over $F_{\mathbf{p}_i}$, and where

$$\varrho : X \times_S \bar{S} \longrightarrow \mathbb{X} \times_{\mathrm{Spec}\, \bar{\kappa}_{\mathbf{p}_i}} \bar{S}$$

is a $O_{K_{\mathbf{p}_i}}$-linear quasi-isogeny such that λ and $\varrho^*(\lambda_{\mathbb{X}})$ differ locally on $\bar{S}$ by a scalar in $\mathbb{Q}_p^\times$. Here $\bar{S}$ denotes the special fiber of S. It is also assumed that the kernel of λ is trivial if $\mathbf{p}_i$ is split, and is of height $f_{\mathbf{q}_i} h_{\mathbf{q}_i}$ and contained in $X[\iota(\mathbf{q}_i)]$ if $\mathbf{p}_i$ has a unique prime ideal $\mathbf{q}_i$ over it. The height of ϱ is a locally constant function on S of the form

$$s \mapsto f_{\mathbf{q}_i} \cdot \check{c}_i(s)\,,$$

where $\check{c}_i : S \longrightarrow \mathbb{Z}$, cf. [26], Lemma 3.53.

Then it is easily seen that $\breve{\mathcal{M}}$ is the formal subscheme of

$$(\breve{\mathcal{M}}_1 \times_{\operatorname{Spf} O_{\breve{E}_1}} \operatorname{Spf} O_{\breve{E}_\nu}) \times_{\operatorname{Spf} O_{\breve{E}_\nu}} \ldots \times_{\operatorname{Spf} O_{\breve{E}_\nu}} (\breve{\mathcal{M}}_{r+s+t} \times_{\operatorname{Spf} O_{\breve{E}_{r+s+t}}} \operatorname{Spf} O_{\breve{E}_\nu})$$

where the functions $\breve{c}_i$ agree.

We now describe the formal schemes $\mathcal{M}_i$ in more detail. For i with $r+1 \leq i \leq r+s+t$, this has been done in [26], 6.46 and 6.48. We record this result as follows.

For $i = r+1, \ldots, r+s$, the formal scheme $\breve{\mathcal{M}}_i$ is the constant étale scheme $G'_{\mathbf{p}_i}(\mathbb{Q}_p)/C_{\mathbf{p}_i}$ for a certain maximal compact subgroup $C_{\mathbf{p}_i}$. In these cases $E_i = \mathbb{Q}_p$, and the Weil descent datum on $\breve{\mathcal{M}}_i \simeq G'_{\mathbf{p}_i}(\mathbb{Q}_p)/C_{\mathbf{p}_i}$ is given by multiplication by a central element $t_i \in G'_{\mathbf{p}_i}(\mathbb{Q}_p)$ (equal to $(1,p)$ in the notation cf. [26], Lemma 6.47). We note that in loc. cit. only the case $h = 0$ is considered; however, the result is also valid in the case $h = 1$, with the same proof.

For $i = r + s + 1, \ldots, r + s + t$, the formal scheme $\breve{\mathcal{M}}_i$ is again the constant étale scheme $G'_{\mathbf{p}_i}(\mathbb{Q}_p)/C_{\mathbf{p}_i}$ for a certain maximal compact subgroup $C_{\mathbf{p}_i}$. In these cases, E_i is the unramified extension of degree u of $\mathbb{Q}_p$, where u is an *even* divisor of $2 \cdot f_{\mathbf{p}_i}$. The Weil descent datum on $\breve{\mathcal{M}}_i \simeq G'_{\mathbf{p}_i}(\mathbb{Q}_p)/C_{\mathbf{p}_i}$ is given by multiplication by a central element $t_i \in G'_{\mathbf{p}_i}(\mathbb{Q}_p)$ (equal to $p^{u/2}$ in the notation of [26], 6.48.)

Now let us consider the uniformizing primes of the first kind, so $1 \leq i \leq r$. Then $G'_{\mathbf{p}_i}$ is the group of unitary similitudes for the hermitian vector space $V_{\mathbf{p}_i}$ of dimension 2 over the quadratic extension $K_{\mathbf{p}_i}$ of $\mathbb{Q}_p$, and $E_i = \mathbb{Q}_p$. The height of the quasi-isogeny ϱ in the quadruple $(X, \iota, \lambda, \varrho)$ defines the function

$$\breve{c}_i : \breve{\mathcal{M}}_i \longrightarrow \mathbb{Z}\,.$$

It is easily seen that $\operatorname{Im}(\breve{c}_i) = 2\mathbb{Z}$.

Let $\mathcal{M}_0$ be the Drinfeld moduli scheme. We recall its definition. Let B be the quaternion division algebra over $\mathbb{Q}_p$, and denote by O_B its maximal order. Then $\mathcal{M}_0$ is the formal scheme $\breve{\mathcal{M}}_0$ over $\operatorname{Spf} \breve{\mathbb{Z}}_p$ with its Weil descent datum to $\operatorname{Spf} \mathbb{Z}_p$ (cf. [26], Theorem 3.72), where the value of $\breve{\mathcal{M}}_0$ on a scheme $S \in \operatorname{Nilp}_{\breve{\mathbb{Z}}_p}$ is the set of isomorphism classes of triples (X, ι_B, ϱ), where (X, ι_B) is a special formal O_B-module on S and where $\varrho : X \times_S \bar{S} \to \mathbb{X} \times_{\operatorname{Spf} \bar{\mathbb{F}}_p} \bar{S}$ is a O_B-linear quasi-isogeny. Here $(\mathbb{X}, \iota_B)$ is a fixed *framing object* over $\bar{\mathbb{F}}_p$. Again the height of ϱ defines a function

$$\breve{c}_0 : \breve{\mathcal{M}}_0 \longrightarrow \mathbb{Z}\,,$$

with $\operatorname{Im}(\breve{c}_0) = 2\mathbb{Z}$. For $n \in \mathbb{Z}$, let

$$\breve{\mathcal{M}}_i[n] = \breve{c}_i^{-1}(2n)\,, \quad i = 0, 1, \ldots, r\,. \tag{6.4}$$

Then $\breve{\mathcal{M}}_0[0] \simeq \breve{\Omega}^2_{\mathbb{Q}_p}$, where $\breve{\Omega}^2_{\mathbb{Q}_p} = \widehat{\Omega}^2_{\mathbb{Q}_p} \times_{\operatorname{Spf} \mathbb{Z}_p} \operatorname{Spf} \breve{\mathbb{Z}}_p$ (*Drinfeld's isomorphism*). Furthermore

$$\breve{\mathcal{M}}_0 \simeq \breve{\mathcal{M}}_0[0] \times \mathbb{Z}\,,$$

via the isomorphisms $\breve{\mathcal{M}}_0[0] \to \breve{\mathcal{M}}_0[n]$ for any n, given by

$$(X, \iota_B, \varrho) \longmapsto (X, \iota_B \circ \operatorname{int}(\Pi)^{-n}, \iota_B(\Pi)^n \circ \varrho)$$

in the notation of loc. cit. Under the isomorphism $\breve{\mathcal{M}}_0 \simeq \breve{\Omega}^2_{\mathbb{Q}_p} \times \mathbb{Z}$, the Weil descent datum on the LHS is given on the RHS by the composite of the natural descent datum on $\breve{\Omega}^2_{\mathbb{Q}_p}$ and translation by 1, cf. [26], Theorem 3.72.

For $1 \leq i \leq r$, we embed $K_{\mathbf{p}_i}$ into B. More precisely,

a) when $K_{\mathbf{p}_i}/\mathbb{Q}_p$ is unramified, we write $O_{K_{\mathbf{p}_i}} = \mathbb{Z}_p[\delta]$, where $\delta^2 \in \mathbb{Z}_p^\times$ and we choose the uniformizer Π of O_B such that Π normalizes $K_{\mathbf{p}_i}$ and satisfies $\Pi^2 = p$.

b) when $K_{\mathbf{p}_i}/\mathbb{Q}_p$ is ramified, we assume $p \neq 2$. Then we choose for Π a uniformizer of $O_{K_{\mathbf{p}_i}}$ such that $\Pi^2 \in \mathbb{Z}_p$.

In [16] we define for any i with $1 \leq i \leq r$ an isomorphism

$$\breve{\mathcal{M}}_0[0] \longrightarrow \breve{\mathcal{M}}_i[0]\,. \tag{6.5}$$

More precisely, we associate to the framing object $(\mathbb{X}, \iota_{\mathbb{X}})$ of $\mathcal{M}_0$ a framing object $(\mathbb{X}, \iota, \lambda_{\mathbb{X}})$ of $\mathcal{M}_i$, where ι is the restriction from O_B to $O_{K_{\mathbf{p}_i}}$ of $\iota_{\mathbb{X}}$, and where $\lambda_{\mathbb{X}}$ is a carefully chosen quasi-polarization of $\mathbb{X}$. Similarly, we associate to any point (X, ι_B, ϱ) of $\breve{\mathcal{M}}_0[0]$ the point $(X, \iota, \lambda_X, \varrho)$ of $\breve{\mathcal{M}}_i[0]$, where $\iota = \iota_B | O_{K_{\mathbf{p}_i}}$ and where λ_X is a carefully chosen quasi-polarization of X.

Lemma 6.5. *The isomorphism* (6.5) *extends to an isomorphism compatible with the Weil descent data,*

$$\breve{\mathcal{M}}_0 \longrightarrow \breve{\mathcal{M}}_i\,. \tag{6.6}$$

Proof. Let $n \in 2\mathbb{Z}$ and let $(X, \iota_B \circ \mathrm{int}(\Pi^{-n}), \iota_{\mathbb{X}}(\Pi)^n \circ \varrho)$ be a point of $\breve{\mathcal{M}}_0[n]$, where $(X, \iota_B, \varrho) \in \breve{\mathcal{M}}_0[0]$. If $K_{\mathbf{p}_i}/\mathbb{Q}_p$ is ramified, we map this point to $(X, \iota_X, \lambda_X, \iota_{\mathbb{X}}(\Pi)^n \circ \varrho) \in \breve{\mathcal{M}}_i[n]$. This makes sense since in this case

$$\iota_{\mathbb{X}}(\Pi)^*(\lambda_{\mathbb{X}}) = -p \cdot \lambda_{\mathbb{X}}\,.$$

If $K_{\mathbf{p}_i}/\mathbb{Q}_p$ is unramified, we map this point to $(X, \iota_X, \lambda_X, \iota_{\mathbb{X}}(\Pi\delta)^n \circ \varrho) \in \breve{\mathcal{M}}_i[n]$. This makes sense since in this case

$$\iota_{\mathbb{X}}(\Pi\delta)^*(\lambda_{\mathbb{X}}) = \mathrm{Nm}\,(\delta) \cdot p \cdot \lambda_{\mathbb{X}}\,.$$

It follows from the definitions that this defines an isomorphism compatible with the descent data on $\breve{\mathcal{M}}_0$ and $\breve{\mathcal{M}}_i$. $\square$

Corollary 6.6. *Let i with $1 \leq i \leq r$. Assume $p \neq 2$ if $\mathbf{p}_i$ is ramified in K. There is an isomorphism*

$$\breve{\mathcal{M}}_i \simeq \breve{\Omega}^2_{\mathbb{Q}_p} \times \mathbb{Z}\,,$$

such that the Weil descent datum on the LHS corresponds to the composite of the natural descent datum on $\breve{\Omega}^2_{\mathbb{Q}_p}$ and translation by 1. $\square$

We leave it to the reader to check that (again for i with $1 \leq i \leq r$) the group $J_i(\mathbb{Q}_p)$ of $O_{K_{\mathbf{p}_i}}$-linear self-isogenies of $\mathbb{X}$ which preserve $\lambda_{\mathbb{X}}$ up to a scalar in $\mathbb{Q}_p^\times$ can be identified with the group of unitary similitudes of the *split* hermitian space of dimension 2 over $K_{\mathbf{p}_i}$, and that the action of an element $g \in J_i(\mathbb{Q}_p)$ on the LHS of Corollary 6.6, given by $(X, \iota, \lambda, \rho) \mapsto (X, \iota, \lambda, g \circ \rho)$, is given on the RHS by

$$(g_{\mathrm{ad}}\,, \text{translation by } \tfrac{1}{2}\,\mathrm{ord}\, c(g))\,,$$

where g_{ad} is considered as an element in $\mathrm{PGL}_2(\mathbb{Q}_p)$, via a chosen isomorphism $(J_i)_{\mathrm{ad}}(\mathbb{Q}_p) = \mathrm{PGL}_2(\mathbb{Q}_p)$.

We refer to [26], 6.46, resp. 6.48 for a description of the analogous groups $J_i(\mathbb{Q}_p)$ for i with $r+1 \leq i \leq r+s$, resp. $r+s+1 \leq i \leq r+s+t$. Let $J(\mathbb{Q}_p)$ be the group of automorphisms of the isocrystal N which are K-linear and preserve the anti-symmetric form up to a scalar in $\mathbb{Q}_p$.

As in [26], Proposition 6.49, one checks the following proposition.

Proposition 6.7. *$J(\mathbb{Q}_p)$ is the inverse image of the diagonal under the map*

$$\prod c_i : \prod_{i=1}^{r+s+t} J_i(\mathbb{Q}_p) \longrightarrow \prod_{i=1}^{r+s+t} \mathbb{Q}_p^\times\,.$$

Similarly, $G(\mathbb{Q}_p)$ is the inverse image of the diagonal under

$$\prod c_i : \prod_{i=1}^{r+s+t} G'_{\mathbf{p}_i}(\mathbb{Q}_p) \longrightarrow \prod_{i=1}^{r+s+t} \mathbb{Q}_p^\times\,.$$

For $1 \leq i \leq r$, let $C_{\mathbf{p}_i}$ be the unique maximal compact subgroup of $G'_{\mathbf{p}_i}(\mathbb{Q}_p)$, and, for $r+1 \leq i \leq r+s+t$, let $C_{\mathbf{p}_i}$ be the maximal compact subgroup of $G'_{\mathbf{p}_i}(\mathbb{Q}_p)$ introduced above. The actions of $J_i(\mathbb{Q}_p)$ on $G'_{\mathbf{p}_i}(\mathbb{Q}_p)/C_{\mathbf{p}_i}$ combine to give an action of $J(\mathbb{Q}_p)$ on $G(\mathbb{Q}_p)/C_p$, where $C_p = G(\mathbb{Q}_p) \cap \prod C_{\mathbf{p}_i}$. In case $p = 2$, assume that $\mathbf{p}_i$ is unramified in K, for $1 \leq i \leq r$.

There is a $J(\mathbb{Q}_p)$-equivariant isomorphism of formal schemes

$$\breve{\mathcal{M}} \simeq \big(\prod_{i=1}^{r} \breve{\Omega}^2_{\mathbb{Q}_p}\big) \times G(\mathbb{Q}_p)/C_p.$$

The action of $J(\mathbb{Q}_p)$ on the first r factors is via the projections $J(\mathbb{Q}_p) \longrightarrow J_i(\mathbb{Q}_p) \longrightarrow \mathrm{PGL}_2(\mathbb{Q}_p)$, and on the last factor is as described above.

The localization E_ν of the reflex field E is the composite of the fields E_i, for $i = r+1, \ldots, r+s+t$ described above. The Weil descent datum on $\breve{\mathcal{M}}$ relative to $\breve{E}_\nu/E_\nu$ induces on the RHS the natural descent datum on the first r factors multiplied with the action of an element $g \in G(\mathbb{Q}_p)$ on the last factor, where g maps to the element $(t_1, \ldots, t_{r+s+t}) \in \prod G'_{\mathbf{p}_i}(\mathbb{Q}_p)$, where $t_{r+1}, \ldots, t_{r+s+t}$ are the central elements described above, and where for $i = 1, \ldots, r$ the element t_i is any element with $\mathrm{ord}\, c_i(t_i) = 1$. □

Using this proposition, we obtain as in [26] the following theorem. When $p = 2$, we make the usual assumption on prime ideals $\mathbf{p}|p$ of the first kind.

Theorem 6.8. *Let $C = C^p \cdot C_p$, where C_p is defined in Proposition 6.7. There is a $G(\mathbb{A}^{\infty,p})$-equivariant isomorphism of formal schemes over $O_{\breve{E}_\nu} = \breve{\mathbb{Z}}_p$,*

$$I(\mathbb{Q})\backslash[(\prod_{i=1}^{r} \breve{\Omega}^2_{\mathbb{Q}_p}) \times G(\mathbb{A}^\infty)/C] \simeq \mathcal{M}_{r,h,V}(C^p)^\wedge \times_{\mathrm{Spf} O_{E_\nu}} \mathrm{Spf} O_{\breve{E}_\nu}.$$

The group I is the inner form of G, unique up to isomorphism, such that $I_{\mathrm{ad}}(\mathbb{R})$ is compact, and $I(\mathbb{Q}_p)$ is the group $J(\mathbb{Q}_p)$ defined above and such that $I(\mathbb{A}^{\infty,p}) \simeq G(\mathbb{A}^{\infty,p})$. The natural descent datum on the RHS induces on the LHS the composite of the natural descent datum on the first r factors multiplied with the action of the element $g \in G(\mathbb{Q}_p) \subset G(\mathbb{A}^\infty)$ in Proposition 6.7 on the last factor. □

Corollary 6.9. *Under the conditions of the previous theorem, $\mathcal{M}_{r,h,V}(C^p) \otimes_{O_{E(p)}} O_{E_\nu}$ is flat over* $\mathrm{Spec}\, O_{E_\nu}$.

Proof. Indeed, flatness holds for the LHS in the last theorem. □

As a special case of the previous theorem, we formulate the following corollary.

Corollary 6.10. *Let K/F be a CM-extension of the totally real field F of degree d over $\mathbb{Q}$. Let p be a prime number that decomposes completely in F and such that each prime divisor $\mathbf{p}$ of p is inert or ramified in K. We also assume that if $p = 2$, then no such $\mathbf{p}$ is ramified. Let V be a hermitian vector space of dimension 2 over K with signature $(1, 1)$ at every archimedean place of K. We also assume that $\mathrm{inv}_{\mathbf{p}}(V) = -1$, for all $\mathbf{p}|p$.*

Let G be the group of unitary similitudes of V with multiplier in $\mathbb{Q}$. Let C^p be an open compact subgroup of $G(\mathbb{A}^{\infty,p})$, and let $C = C^p \cdot C_p$, where C_p is the unique maximal compact subgroup of $G(\mathbb{Q}_p)$. Let Sh_C be the corresponding Shimura variety, which is defined over $\mathbb{Q}$.

Let $\mathcal{M}_{r,h,V}(C^p)$ be the model of Sh_C over $\mathbb{Z}_{(p)}$ which parametrizes almost principal CM-triples (A, ι, λ) of type (r, h) with level-C^p-structure, where $r_\varphi = 1$, for all φ, and with $\mathrm{inv}_v(A, \iota, \lambda) = \mathrm{inv}_v(V)$, for all v. We assume for any prime divisor $\mathbf{p}$ of p that $h_{\mathbf{p}} = 0$, resp. $h_{\mathbf{p}} = 1$, when $\mathbf{p}$ is ramified, resp. inert. We also assume the following compatibility condition between h and the invariants of V, cf. Proposition 4.2:

- *If $h_{\mathbf{p}_v} = 0$ and v is inert in K/F, then $\mathrm{inv}_v(V) = 1$.*

- *If* $h_{\mathbf{p}_v} = 2$*, then* $\mathrm{inv}_v(V) = 1$.
- *If* $h_{\mathbf{p}_v} = 1$*, then* v *is inert in* K/F *and* $\mathrm{inv}_v(V) = -1$.

Then there is a $G(\mathbb{A}^{\infty,p})$*-equivariant isomorphism of* p*-adic formal schemes*

$$I(\mathbb{Q})\backslash\big[\big((\widehat{\Omega}^2_{\mathbb{Q}_p})^d \times_{\mathrm{Spf}\,\mathbb{Z}_p} \mathrm{Spf}\,\breve{\mathbb{Z}}_p\big) \times G(\mathbb{A}^\infty)/C\big] \simeq \mathcal{M}_{r,h,V}(C^p)^\wedge \times_{\mathrm{Spf}\,\mathbb{Z}_p} \mathrm{Spf}\,\breve{\mathbb{Z}}_p\,.$$

Here $I(\mathbb{Q})$ *is the group of* $\mathbb{Q}$*-rational points of the inner form* I *of* G *such that* $I_{\mathrm{ad}}(\mathbb{R})$ *is compact, and* $I_{\mathrm{ad}}(\mathbb{Q}_p) \simeq \mathrm{PGL}_2(\mathbb{Q}_p)^d$*, and* $I(\mathbb{A}^{\infty,p}) \simeq G(\mathbb{A}^{\infty,p})$.

The natural descent datum on the RHS induces on the LHS the natural descent datum on the first factor multiplied with the translation action of $(1,t)$ *on* $G(\mathbb{A}^{\infty,p})/C^p \times G(\mathbb{Q}_p)/C_p$*, where* $t \in G(\mathbb{Q}_p)$ *is any element with* $\mathrm{ord}\,c(t) = 1$. □

We end this section by showing an integral version of Theorem 1.1 in the introduction. It shows that in the special case $F = \mathbb{Q}$, the theory of local invariants of CM-triples of section 3 is not needed. We use the notation introduced before the statement of Theorem 1.1. We consider the DM-stack $\mathcal{M}_h(C^p)$ over $\mathrm{Spec}\,\mathbb{Z}_{(p)}$ which parametrizes quadruples (A,ι,λ,η^p), where (A,ι,λ) is an almost principal CM-triple of type (r,h) where $r_\varphi = 1$ for all φ, and where h satisfies the usual compatibility condition of Proposition 4.2 and such that $h_p = 1$, resp. $h_p = 0$, according as p is unramified or ramified in the quadratic extension K of $\mathbb{Q}$. Finally, η^p is a level-C^p-structure. The next theorem implies Theorem 1.1. We make the usual assumption when $p = 2$.

Theorem 6.11. *There are canonical isomorphisms between schemes over* $\mathrm{Spec}\,\mathbb{Q}$*,*

$$\mathcal{M}_h(C^p) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \simeq \mathrm{Sh}_C,$$

and between schemes over $\mathrm{Spec}\,\breve{\mathbb{Z}}_p$*,*

$$\mathcal{M}_h(C^p) \otimes_{\mathbb{Z}_{(p)}} \breve{\mathbb{Z}}_p \simeq \big(\bar{G}(\mathbb{Q})\backslash[\widehat{\Omega}^2_{\mathbb{Q}_p} \times G(\mathbb{A}^\infty)/C]\big) \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p.$$

Proof. All we have to prove is that $\mathcal{M}_h(C^p) = \mathcal{M}_{r,h,V}(C^p)$, i.e., that $\mathrm{inv}_v(A,\iota,\lambda) = \mathrm{inv}_v(V)$, $\forall v$. This is obvious if $\mathrm{char}\,k = 0$, because $\mathrm{inv}_v(A,\iota,\lambda) = \mathrm{inv}_v(V)$ for all places $v \neq p$ of $\mathbb{Q}$ by the existence of the level structure η^p, and by the product formula for $\mathrm{inv}_v(A,\iota,\lambda)$ and for $\mathrm{inv}_v(V)$, cf. Proposition 3.4, (i). Now let $\mathrm{char}\,k = p$. But the moduli space $\mathcal{M}_h(C^p)$ is flat over $\mathrm{Spec}\,\mathbb{Z}_{(p)}$, as follows from the theory of local models. Indeed, the ramified case is covered by the theorem of Pappas, [20], Thm. 4.5., b), cf. also [21], Remark 2.35. The unramified case is covered by the theorem of Görtz, cf. [9], Thm. 4.25, cf. also [21], Thm. 2.16. Hence we may apply Proposition 3.4, (ii) to deduce that the product formula for $\mathrm{inv}_v(A,\iota,\lambda)$ is also valid in this case, and hence the same argument implies the claim. □

Remark 6.12. It is the flatness issue that prevents us from allowing primes $\mathbf{p}$ such that $K_{\mathbf{p}}/F_{\mathbf{p}}$ is ramified in the definition of uniformizing primes of the second kind. Indeed, the naive local models are known to be non-flat in this situation, cf. [21]. By imposing more conditions on (A,ι,λ), it should be possible to formulate a moduli problem that is flat; this would yield more general situations in which integral uniformization holds.

7. Rigid-analytic uniformization

In this section, we will consider rigid-analytic uniformization. This is weaker than integral uniformization, but more general in two ways. First, since flatness of integral models is no issue, we are able to also allow *degenerate* CM-types at ramified primes above p, comp. Remark 6.12. Secondly, we can allow for level of the form $C = C^pC_p$, where C_p can be strictly smaller than a maximal compact subgroup of $G(\mathbb{Q}_p)$. The latter variant is inspired by the corresponding rigid-analytic uniformization theorem of Drinfeld, cf. [2, 8].

We first formulate a rigid-analytic version of Theorem 6.8. We now allow also primes of the fourth kind, extending the list from Definition 5.3.

Definition 7.1. Let K/F be a *ramified* quadratic extension of p-adic fields. We call $(K/F, r, h, \varepsilon)$ *uniformizing data of the fourth kind*, if $r_\varphi \in \{0, 2\}$ for all $\varphi \in \mathrm{Hom}(K, \bar{\mathbb{Q}}_p)$, and $h = 0$ and $\varepsilon = \pm 1$.

In the notation of section 6, we continue the enumeration of the prime ideals of F over p by also allowing prime ideals $\mathbf{p}_{r+t+s+1}, \ldots, \mathbf{p}_{r+s+t+u}$ of the fourth kind. Then the rigid-analytic space $\breve{\mathcal{M}}^{\mathrm{rig}}$ associated to $\breve{\mathcal{M}}$ is the subspace of

$$(\breve{\mathcal{M}}_1^{\mathrm{rig}} \times_{\mathrm{Sp}\, \breve{E}_1} \mathrm{Sp}\, \breve{E}_\nu) \times_{\mathrm{Sp}\, \breve{E}_\nu} \ldots \times_{\mathrm{Sp}\, \breve{E}_\nu} (\breve{\mathcal{M}}_{r+s+t+u}^{\mathrm{rig}} \times_{\mathrm{Sp}\, \breve{E}_{r+s+t+u}} \mathrm{Sp}\, \breve{E}_\nu)$$

where the functions $\breve{c}_i$ agree. Now since the CM-type for primes of the fourth kind is degenerate, all points of the rigid spaces $\breve{\mathcal{M}}_i^{\mathrm{rig}}$ map to the same point under the *period map*, for $i = r+s+t+1, \ldots, r+s+t+u$, cf. [26]. Therefore, these points all lie in one single isogeny class, and are classified by their p-adic Tate module. The set of these Tate modules is a homogeneous space under $G'_{\mathbf{p}_i}(\mathbb{Q}_p)$. In fact, denoting by $C_{\mathbf{p}_i}$ the stabiliser of a self-dual lattice in $V_{\mathbf{p}_i}$, we can identify $\breve{\mathcal{M}}_i^{\mathrm{rig}}$ with the discrete space $G'_{\mathbf{p}_i}(\mathbb{Q}_p)/C_{\mathbf{p}_i}$. Furthermore, $J_i(\mathbb{Q}_p) = G'_{\mathbf{p}_i}(\mathbb{Q}_p)$. Let Φ_i be the subset of $\mathrm{Hom}_{\mathbb{Q}_p}(K_{\mathbf{p}_i}, \bar{\mathbb{Q}}_p)$ where the value of r_i is equal to zero. The reflex field E_i is characterized by

$$\mathrm{Gal}(\bar{\mathbb{Q}}_p/E_i) = \{\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \mid \sigma(\Phi_i) = \Phi_i\}\,.$$

The Weil descent datum is given by translation by an element in the center of $G'_{\mathbf{p}_i}(\mathbb{Q}_p)$ on $G'_{\mathbf{p}_i}(\mathbb{Q}_p)/C_{\mathbf{p}_i}$. This central element can be deduced from the reciprocity map attached to $G'_{\mathbf{p}_i}$ and r_i.

We may now formulate the following analogue of Theorem 6.8. The assumptions are the same, except that we now also accept uniformizing primes of the fourth kind. The proof is completely analogous.

Theorem 7.2. *Let $C = C^p \cdot C_p$, where $C_p = G(\mathbb{Q}_p) \cap \prod_{i=1}^{r+s+t+u} C_{\mathbf{p}_i}$, and define $J(\mathbb{Q}_p)$ and its action on $G(\mathbb{Q}_p)/C_p$ in analogy with Proposition 6.7. In case $p = 2$, assume that p is unramified in $K_{\mathbf{p}_i}$, for $1 \le i \le r$. There is a $G(\mathbb{A}^{\infty,p})$-equivariant isomorphism of rigid-analytic spaces over $\breve{E}_\nu$,*

$$I(\mathbb{Q})\backslash\big[\big(\prod_{i=1}^{r} \Omega^2_{\mathbb{Q}_p} \times_{\mathrm{Sp}\,\mathbb{Q}_p} \mathrm{Sp}\, \breve{E}_\nu\big) \times G(\mathbb{A}^\infty)/C\big] \simeq \mathcal{M}_{r,h,V}(C^p)^{\mathrm{rig}} \times_{\mathrm{Sp}\, E_\nu} \mathrm{Sp}\, \breve{E}_\nu.$$

The group I is the inner form of G, unique up to isomorphism, such that $I_{\mathrm{ad}}(\mathbb{R})$ is compact, and $I(\mathbb{Q}_p)$ is the group $J(\mathbb{Q}_p)$ defined above and such that $I(\mathbb{A}^{\infty,p}) \simeq G(\mathbb{A}^{\infty,p})$. The natural descent datum on the RHS induces on the LHS the composite of the natural descent datum on the first r factors multiplied with the action of an element $g \in G(\mathbb{Q}_p) \subset G(\mathbb{A}^\infty)$, described in Proposition 6.7 and above, on the last factor. $\square$

As a special case, we formulate a corollary in the style of Theorem 1.1.

Corollary 7.3. *Let p decompose completely in the totally real field F of degree d over $\mathbb{Q}$, and let K/F be a CM quadratic extension such that each prime divisor $\mathbf{p}$ of p in F is inert or ramified in K. We also assume that if $p = 2$, then no $\mathbf{p}$ is ramified. Let V be a hermitian vector space of dimension 2 over K with signature (1, 1) at every archimedean place of F. We also assume that $\mathrm{inv}_{\mathbf{p}}(V) = -1$ for all $\mathbf{p}|p$. Let G be the group of unitary similitudes of V with multiplier in $\mathbb{Q}^\times$. Let C^p be an open compact subgroup of $G(\mathbb{A}^{\infty,p})$, and let $C = C^p \cdot C_p$, where C_p is the unique maximal compact subgroup of $G(\mathbb{Q}_p)$. Let Sh_C be the canonical model of the corresponding Shimura variety, a projective variety of dimension d defined over $\mathbb{Q}$.*

There is a $G(\mathbb{A}^{\infty,p})$-equivariant isomorphism of projective schemes over $\breve{\mathbb{Q}}_p$,

$$\mathrm{Sh}_C \otimes_{\mathbb{Q}} \breve{\mathbb{Q}}_p \simeq \big(I(\mathbb{Q})\backslash[(\Omega^2_{\mathbb{Q}_p})^d \times G(\mathbb{A}^\infty)/C]\big) \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p$$

Here $I(\mathbb{Q})$ is the group of $\mathbb{Q}$-rational points of the inner form I of G such that $I_{\mathrm{ad}}(\mathbb{R})$ is compact, $I_{\mathrm{ad}}(\mathbb{Q}_p) \simeq \mathrm{PGL}_2(\mathbb{Q}_p)^d$, and $I(\mathbb{A}^{\infty,p}) \simeq G(\mathbb{A}^{\infty,p})$.

We next allow deeper level structures at p.

We first point out a further variant of the moduli space introduced in Remark 4.5. Namely, denoting by C_M the stabilizer of the fixed lattice M in V, we may introduce, for any open compact subgroup C of $G(\mathbb{A}^\infty)$ which is contained with finite index in C_M, the stack $\mathcal{M}_{r,h,V}(C)$ over $\operatorname{Spec} E$ which, in addition to (A, ι, λ) satisfying conditions (4.2) and (4.3), fixes a level structure $\operatorname{mod} C$, i.e., an isomorphism compatible with ι and λ

$$\widehat{T}(A) \simeq M \otimes \widehat{\mathbb{Z}} \operatorname{mod} C\,,$$

in the sense of Kottwitz [14]. Note that, due to the existence of the level structure, the condition (4.3) is automatic. If $C = C^p C_p^0$, where C_p^0 denotes the maximal compact subgroup of $G(\mathbb{Q}_p)$ occurring in the previous theorem, then by Lemma 4.3, $\mathcal{M}_{r,h,V}(C) = \mathcal{M}_{r,h,V}(C^p) \otimes_{O_{E_{(p)}}} E$.

Recall the formal moduli space $\breve{\mathcal{M}}$ over $\operatorname{Spf} O_{\breve{E}_\nu}$ of the beginning of this section; in particular, we allow uniformizing primes of the fourth kind. We denote by (X, ι, λ) the universal object over $\breve{\mathcal{M}}$. Let $\breve{\mathcal{M}}^{\mathrm{rig}}$ be the rigid-analytic space over $\operatorname{Sp} \breve{E}$ associated to the formal scheme $\breve{\mathcal{M}}$. For any open compact subgroup C_p contained in the maximal compact subgroup C_p^0, we may consider the rigid-analytic space $\breve{\mathbb{M}}_{C_p}$ which trivializes the local system $T_p(X)$ on $\breve{\mathcal{M}}^{\mathrm{rig}}$,

$$T_p(X) \simeq M \otimes \mathbb{Z}_p \operatorname{mod} C_p\,.$$

Here M is again a fixed lattice in V, as in Remark 4.5, and the isomorphism is, of course, supposed to be $O_{K\otimes\mathbb{Q}_p}$-linear and to preserve the symplectic forms up to a scalar in $\mathbb{Z}_p^\times$. The rigid space $\breve{\mathbb{M}}_{C_p}$ comes with a Weil descent datum to $\operatorname{Sp} E_\nu$.

Comparing now the p-primary level structures of an abelian variety and of its associated p-divisible group, we obtain the following rigid-analytic uniformization theorem.

Theorem 7.4. *Let C be of the form $C^p C_p$. Then there exists a $G(\mathbb{A}^\infty)$-equivariant isomorphism of rigid-analytic spaces over $\operatorname{Sp} \breve{E}_\nu$, compatible with the Weil descent data on both sides,*

$$I(\mathbb{Q}) \backslash \breve{\mathbb{M}}_{C_p} \times G(\mathbb{A}^{\infty,p})/C^p \simeq \mathcal{M}_{r,h,V}(C)^{\mathrm{rig}} \times_{\operatorname{Sp} E_\nu} \operatorname{Sp} \breve{E}_\nu\ .$$

□

Again, as for the integral version of the uniformization theorem, one can make this statement more explicit. For this, we suppose that C_p is of the form

$$C_p = G(\mathbb{Q}_p) \cap \prod_{i=1}^{r+s+t+u} C_{\mathbf{p}_i}\,, \tag{7.1}$$

where $C_{\mathbf{p}_i}$ are open compact subgroups in $G'_{\mathbf{p}_i}(\mathbb{Q}_p)$. We consider the corresponding finite etale coverings $\breve{\mathbb{M}}_{C_{\mathbf{p}_i}}$ of $\breve{\mathcal{M}}_i^{\mathrm{rig}}$, for $i = 1, \dots, r+s+t+u$, i.e., those trivializing $\operatorname{mod} C_{\mathbf{p}_i}$ the p-adic Tate modules of the universal objects over $\breve{\mathcal{M}}_i^{\mathrm{rig}}$. For $i = r+1, \dots, r+s+t+u$, $\breve{\mathbb{M}}_{C_{\mathbf{p}_i}}$ is the constant etale scheme $G'_{\mathbf{p}_i}(\mathbb{Q}_p)/C_{\mathbf{p}_i}$, with the Weil descent datum described in the previous section. It is equipped with the obvious morphism to $\mathbb{Q}_p^\times / c_i(C_{\mathbf{p}_i})$.

Now for i with $1 \le i \le r$, and with the usual assumption when $p = 2$, we have an isomorphism

$$\breve{\mathcal{M}}_i \simeq \big(\widehat{\Omega}^2_{\mathbb{Q}_p} \times_{\operatorname{Spf} \mathbb{Z}_p} \operatorname{Spf} \breve{\mathbb{Z}}_p\big) \times \mathbb{Z} = \big(\widehat{\Omega}^2_{\mathbb{Q}_p} \times_{\operatorname{Spf} \mathbb{Z}_p} \operatorname{Spf} \breve{\mathbb{Z}}_p\big) \times G'_{\mathbf{p}_i}(\mathbb{Q}_p)/C^0_{\mathbf{p}_i}$$

as in Corollary 6.6, which induces isomorphisms of rigid-analytic spaces

$$\breve{\mathcal{M}}_i^{\mathrm{rig}} \simeq \big(\Omega^2_{\mathbb{Q}_p} \times_{\operatorname{Sp} \mathbb{Q}_p} \operatorname{Sp} \breve{\mathbb{Q}}_p\big) \times \mathbb{Z} \simeq \big(\Omega^2_{\mathbb{Q}_p} \times_{\operatorname{Sp} \mathbb{Q}_p} \operatorname{Sp} \breve{\mathbb{Q}}_p\big) \times G'_{\mathbf{p}_i}(\mathbb{Q}_p)/C^0_{\mathbf{p}_i}\,. \tag{7.2}$$

The covering space $\breve{\mathbb{M}}_{C_{\mathbf{p}_i}}$ of $\breve{\mathcal{M}}_i^{\mathrm{rig}}$ maps to the discrete space $\mathbb{Q}_p^\times / c_i(C_{\mathbf{p}_i})$, covering the second projection of $\breve{\mathcal{M}}_i^{\mathrm{rig}}$ to $\mathbb{Z} \simeq \mathbb{Q}_p^\times / c_i(C^0_{\mathbf{p}_i})$ in (7.2), comp. also [5].

We now obtain the following description of the covering space $\breve{\mathbb{M}}_{C_p}$, which makes Theorem 7.4 more explicit.

Theorem 7.5. *The covering space $\breve{\mathbb{M}}_{C_p}$ can be identified with the inverse image of $\mathbb{Q}_p^\times/c(C_p) \subset \prod_{i=1}^{r+s+t+u} \mathbb{Q}_p^\times/c_i(C_{\mathbf{p}_i})$ under the morphism of rigid-analytic spaces*

$$\prod_{i=1}^{r} \breve{\mathbb{M}}_{C_{\mathbf{p}_i}} \times \prod_{i=r+1}^{r+s+t+u} G'_{\mathbf{p}_i}(\mathbb{Q}_p)/C_{\mathbf{p}_i} \longrightarrow \prod_{i=1}^{r+s+t+u} \mathbb{Q}_p^\times/c_i(C_{\mathbf{p}_i})\,.$$

□

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